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This is Deliverable 4.4 of the Work Package 4 (WP4) in “Training for Big Data in Financial Research and Risk Management” (BigDataFinance) Innovative Training Network Marie Sklodowska-Curie project 2015-2019.

Name of the deliverable: A report on a tested and validated risk management tool based on scaling laws for FX markets on RP 13

Description: The report is the summary of the research works done as part of the project “High Frequency Trading Risk Management Tools Based on Scaling Laws”. It includes both the high-level and the detailed description of programming tools designed for high-frequency data analysis and risk management. The tools and their practical applications are explained in several research lines attached to the report. One of the included research papers describes how an agent-based model where the trades happen in directional-change intrinsic time can replicate statistical properties observed in empirical data. The short extension of this work demonstrates the ability of the agents to reproduce the position ratio typical for the Forex markets. The second paper describes the instantaneous volatility seasonality of Forex and Bitcoin exchange rates observed in directional-change intrinsic time. Scaling laws found in the Bitcoin/USD and EUR/USD markets are presented as a part of the work in progress. Also, the analytic relationship that connects scaling laws in physical and directional-change intrinsic time is provided in the report.

Date, place: November 28th, 2018, Zurich, Switzerland

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Summary of the report:

This section provides a brief summary of the main achievements made under the Marie Curie project “BigDataFinance” on the topic “Tested and validated risk management tool based on scaling laws for FX markets”. The report describes two submitted papers to peer-reviewed journals as well as their extensions. One working paper under the title “Multidimensional Directional–Change Intrinsic Time” is summarized and provided further in the corresponding section.

A set of research tools developed for risk-management of high-frequency data has been designed using the object-oriented computer-programming language Java and uploaded to the public GitHub repository. The repository can be freely accessed through the following link: https://github.com/VladUZH/VIPetrov/blob/master/README.md. All designed programs have been validated by sophisticated data from the foreign exchange market obtained through the Olsen Group. Three separate folders in the repository contain groups of classes that describe the numeric methods. All files have self-explanatory names and are equipped with detailed descriptions of their main objectives as well as by references to the publicly available literature where these methods have been described and implemented for the first time. The current report contains a set of publications, working papers and notes describing the performance of designed tools as well as their potential applications. The following three paragraphs outline the main research works where the tools have been implemented.

1. There is a wide range of research papers intended to understand the behaviour of real market participants by designing and testing artificial agent-based models with specific attributes. The models primarily rely on the physical time as an input for agent’s activity, thus setting limits on the range of areas where they can be utilised. We developed an event-driven agent-based model that replicates statistical properties typical for high-frequency foreign exchange rates. In the model, the agents react only to the sequence of alternating trend changes of various scales and decide on the direction of their trades. Even the basic version of the model manages to replicate both well-known stylized facts typical for all high-frequency markets, as well as the recently discovered scaling laws found in Forex data. The paper “Agent-Based Model in Directional–Change Intrinsic Time” written together with Anton Golub and Richard Olsen has been submitted to the peer-reviewed journal Quantitative Finance.
2. Drawups and drawdowns defined as price returns of given magnitude are scrutinised in numerous research papers where scientists investigate the size, periodicity and the time of recovery associated with such events. In a second work, we employed the alternating sequence of drawups and drawdowns to express historical price moves in terms of directional-change intrinsic events. The event-based time allows analysis of exchange rates that eliminates disadvantages of the calendar time related to the stochasticity of volatility. It resulted in two new estimators of the market's activity, namely instantaneous and realised volatilities, which significantly differ from classical approaches based on squared price returns measured over fixed time intervals. The new approach has been used to uncover weekly seasonality of volatility typical for high-frequency markets such as Forex and Bitcoin. The work has been submitted to Quantitative Finance with the title “Instantaneous Volatility Seasonality of Bitcoin in Directional-Change Intrinsic Time”, written together with Anton Golub and Richard Olsen.

3. In the third line of work, we extended the concept of one-dimensional intrinsic time to the multidimensional case to analyze multiple financial time series. The new method uncovers scaling laws formed in the set of prices which correspond to simultaneously evolving markets. Scaling laws establish invariance of scale and have been applied to risk management and volatility modeling of such complex systems as high-frequency markets. The same risk-management tools, effectively used in the one-dimensional case, can now be applied to study significantly more detailed information extracted from the multitude of aggregated data. This line of research is covered in a working paper entitled “Multidimensional Directional-Change Intrinsic Time” that is planned to be submitted by the end of 2018.
Stochastic volatility is an omnipresent property of all financial time series. Due to this phenomenon, equidistant time-stamps - synchronized with the flow of physical time and used to make snapshots of the market states - are either (1) too sparse and do not capture all the available high-frequency information or are (2) too dense which results in superfluous noisy events in the final time series. Returns, computed with equal time intervals, might erroneously contribute to the volatility estimation. Nevertheless, the majority of risk management tools persistently refer to physical time as the ultimate measure of financial activity (including the famous Black and Scholes equation). Directional-change intrinsic time [Guillaume et al., 1997] is one of the first endeavours to overcome the stiffness of the traditional approach and provides a new concept devoid of the mentioned shortcomings. Intrinsic time ticks when the price experiences alternating reversals of a fixed threshold $\delta$ from local extremes. This approach is sensitive to the markets' activity and registers more ticks after the significant financial news (periods of soaring volatility) while hardly ticking over weekends. Directional-change intrinsic time was successfully applied to the liquidity estimation problem [Golub et al., 2014] and was instrumental in exploiting a Forex market trading strategy characterized by a Sharp ratio above 3 [Golub et al., 2017]. To the best of our knowledge, in all existing articles on the topic of intrinsic time, only individual time series were analysed. The aim of our work is to extend the definition of directional-change intrinsic time to a multidimensional space, where the dimensions are formed by orthogonally placed exchange rates. The new method is tested on empirical time series from the Forex market. All details required for deriving the new algorithm are presented. Moreover, the exact description of the multidimensional dissection procedure is described in the first part of the article. We uncover two scaling laws (directional change count and overshoot move [Glattfelder et al., 2011]) in the multidimensional space and explain their dependence on the number of time series forming the space.
References


Replicating Properties of Forex Market with Agent-Based Model in Directional-Change Intrinsic Time

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We describe an agent-based model where trades happen in event-based time called directional-change intrinsic time. Events are defined as the reversal price move of a directional-change threshold from the local extreme. The price impact of traded volumes is modelled according to the empirically observed squared root impact function. The generated time series reproduces statistical properties of foreign exchange rates: low auto-correlation of returns, fat-tailed distribution of returns, aggregated normality, and the price jump scaling law. Furthermore, we introduce and use as a benchmark, the overshoot scaling law, which is an omnipresent feature of liquid markets and relates the expected length of price overshoots to the length of the corresponding directional-change threshold.

Keywords: Agent-based model; Stylized facts; Forex; Directional-change; Intrinsic time; Scaling laws

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1. Introduction

According to the Bank for International Settlements, the daily trading volume in Foreign Exchange (FX) markets averaged $5.1 trillion per day in April 2016.[3] This volume is generated by the enormous number of transactions made by individual and institutional traders. The proper understanding of the mechanisms of their impact on the overall financial system is crucial for designing risk management tools capable of foreseeing and efficiently cope with the impact of any political, environmental or technical change pertinent to the health of this global market. Due to the large trading volumes and the multitude of participants, the FX market is one of the biggest financial systems where agent-based models were extensively applied for its analysis (Aloud et al. 2017).

According to the efficient-market hypothesis, the consolidated behaviour of all market participants acting on behalf of their perception of all available information is fully reflected in the prices

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1https://www.bis.org/publ/rpfx16fx.pdf
of the traded financial instrument (Fama 1970). This fact encouraged scientists to look into historical time series to study the attributes of the aggregated activity of all principals involved in the trading. Multiple works have been done on searching for fundamental properties in various financial markets compacted in a large amount of historical data. For example, Bollerslev and Melvin (1994) used more than 300 000 quotes for an empirical analysis of the bid-ask spread and its relation to the exchange rate uncertainty. Danielsson and de Vries (1997) and Dacorogna et al. (2001) used high-frequency data to estimate the fat tails of exchange rate returns. Kozhan and Salmon (2012) employed dataset of submitted market and limit orders to examine how the information contained in order books could be exploited in simple trading schemes.

A wide range of agent-based models was proposed to find the exact relationship between actions of individual groups of agents and the observed market responses on it. The focus of the models is concentrated on the attempts to replicate the evolving behaviour of real market participants by creating artificial agents who would impact the market in the same way it happens in reality (for example, Ehrentreich (2007) analyses whether intelligent agents would converge to the homogeneous rational expectations equilibrium or not). Most of the models represent complex systems, populated by a large number of independent and heterogeneous actors competing with each other (Naciri and Tkouiat 2016). Agent-based models are aimed to reproduce and explain phenomena of real markets, such as bubbles, crashes, and regime switches (Samandidou et al. 2007).

Although there is a wide range of various designs with different specific features intended to imitate a given phenomenon observed in reality, one element is always present in any agent-based model: the definition of time and how the agents should dynamically adapt themselves perceiving its flow. Scientists used to rely on physical time where hours, days or even seasons are selected to measure elapsed periods and intervals between events to describe the internal interaction between agents and the impact of their trading activity on the market prices. However, the real market is a complex system with its endogenous non-constant time stream, the activity of which is dependent on the inhomogeneous frequency of political, social or environmental events (Guillaume et al. 1997). This aspect can be easily understood through the fact of non-constant volatility typical for the FX market: samples of equidistant time intervals of different lengths used to compute volatility can significantly affect its results (Müller et al. 1997). Therefore, financial instruments and agent-based models built upon physical time and designed to measure or replicate statistical properties of real time series are naturally limited by the stiffness of selected equidistant intervals.

The concept of directional-change intrinsic time was proposed by Guillaume et al. (1997) in order to overcome this limitation and to provide a more robust framework. In this representation, events are endogenously defined as reversal price moves of a directional-change threshold from local extremes. Applying this measure to a real price curve one dissect it in the sequence of alternating ascending and descending trends. The change in the previous trend is indicated by directional change prices located on the selected fixed distance from the highest or lowest prices of the lastly observed trend. Continuous price moves in the same direction as the registered reversal are called overshoots and represent trend components of the price curve. Only such price moves allow the system’s clock tick. Thus, this intrinsic time is capable of dealing with the essential properties of price curves such as trends independently of the time intervals chosen to observe them. In other words, intrinsic time is sovereign of the scale and speed of the physical time.

Trend changes are the primary indicators of the prevailed side of the aggregated behaviours of all market participants. The proposed in our work agent-based model uses the directional-change intrinsic time framework to monitor trends and dissect price curve into a collection of directional-change points. These points and price overshoots used by the agents as signals for performing trades. Agents, operating in directional-change intrinsic time, employ prices as the only source of information to judge the market conditions and to make decisions about their actions.

A successful agent-based model should be able to reproduce statistical properties of real financial markets. These properties are mostly known as ‘stylized facts’ and are copiously discussed in numerous papers (Kaldor 1961, Pagan 1996, Kullmann et al. 1999, Gençay et al. 2001, Chakraborti et al. 2009). In this work, we show how the synthetically created by the model time series are tested.
by a set of benchmarks to reveal embedded statistical properties. Among the tests are the following stylized facts: low autocorrelation of returns, fat-tailed distribution of returns, aggregational normality, the price jumps scaling law\(^1\) and the overshoot scaling law. The latter statistical property of the FX market is the new way of the testing agent-based models and is chosen in addition to the previous four well-known attributes usually adopted as benchmarks for this purpose. This scaling law was recently found in a wide range of real high-frequency time series and even in arithmetic Brownian motion (Glattfelder et al. 2011). It establishes a relation between the average length of overshoot and the corresponding size of the directional-change threshold and says that on average, a directional change is followed by an overshoot of the same magnitude. The sovereignty of the directional-change concept of the flow of physical time makes this property a very convenient tool for testing agent-based models. To the extent of our knowledge, it is the first research work where this stylized fact based on the directional-change intrinsic time is used to evaluate time series generated by a group of heterogeneous agents.

The outline of the remaining paper is as follows. Section 2 illustrates the directional-change intrinsic time framework in details and provides an example of a real price curve dissected into a collection of intrinsic events. Section 3 describes two main components of the agent-based model: the set of artificial agents and the market impact function. In Section 4 the collection of benchmarks used to validate the properties of the generated time series is discussed. In Section 5 we present all obtained results and observed statistical properties. In Appendix A the average length of overshoots is derived for the case of Brownian motion with the constant trend. In Appendix B the pseudo-code of the directional-change intrinsic time algorithm is provided. Appendix C concludes the paper with a collection of graphics describing all auxiliary experiments.

2. Intrinsic Time

The number of performed transactions as well as traded volumes in liquid markets is much lower during holidays or weekends than during working days or after some unexpected but significant news. This evidence directly contributes to the continuous changes in the volatility of financial time series over time (Blattberg and Gonedes 1974, Christie 1982, Scott 1987). Thanks to this non-homogeneous nature of markets, one can find drastically distinct price patterns inside two separate historical time intervals of the same length. One period could be characterised by an instant price drop by several percents and not less instant recovery to the same level (a flash crash) when the other could represent an absolute standstill. The latter often happens when the market is completely inactive due to holidays season. Despite this well-known fact the historical dynamics of financial markets have been mostly analysed using snapshots of markets’ states equally spaced in time. Mandelbrot and Taylor (1967) and Clark (1973) were one of the first researchers to propose an alternative event-based paradigm for modelling and analysing financial time series. Later, Guillaume et al. (1997) extended the set of available techniques by introducing the directional-change intrinsic time where ticks happen as a result of alternating rising and falling price moves of a certain size (threshold). The choice of the threshold affects the scale of the trend captured by the directional-change intrinsic time. This is especially important taking into account that the same price curve can be characterised by different directions of short- and long-term trends. This phenomenon has frequently been exploited by traders and one specific trading strategy is called ‘Triple Screen Trading System’ was proposed by Elder (2014).

The purpose of the directional-change intrinsic time is to register moments at which the price curve alternates its trend on the given scale and to find extreme prices correspond to maximum or minimum between two consecutive turns. The next paragraph contains the detailed description of the dissection procedure. An example of a real price curve dissected by the intrinsic time is shown in figure 1.

\(^1\)Scaling law (power law): the mathematical relationship between two variables that holds true over multiple orders of magnitude.
To initialise the dissection of a price curve into a set of directional changes, the latest price should be taken as the starting point. Then, the relative size of the directional-change thresholds \( \delta \) and the initial course of the trend (for each given thresholds there are only two possible states: \text{mode}_{\text{up}} \) for the upward or \text{mode}_{\text{down}} \) for the downward trend) should be selected. Each new tick has to be compared to the latest registered extreme \( S_{\text{ext}} \) initially equal to the latest price. If the current mode is \text{mode}_{\text{up}} \) (\text{mode}_{\text{down}} \) and the newest price \( S_{\text{tick}} \) is higher (lower) than the extreme price \( S_{\text{ext}} \), then the \( S_{\text{ext}} \) takes the value of \( S_{\text{tick}} \). Alternatively, the distance between the latest price \( S_{\text{tick}} \) and the current local extreme \( S_{\text{ext}} \) should be compared to the size of the threshold \( \delta \). If the distance is bigger or equal to \( \delta \) then the current price is a new directional-change point. At this moment the \text{mode} \) should be changed to the opposite one and the local extreme reinitialized by \( S_{\text{ext}} \leftarrow S_{\text{tick}} \). If the price curve continues its path in the direction of the trend observed before the latest directional-change event, the overshoot part of the price curve begins. In the work Golub \textit{et al.} (2017) overshoot intrinsic events are registered every time when the size of the overshoot is a multiple of the dissection threshold \( \delta \). This additional category of intrinsic events was not described in the original paper and is introduced here to connect the empirically observed length of the overshoot section with the decision-making mechanism of the agent-based model. The details on the expected length will be provided in the section 4.2 where the overshoot scaling law is described.

There is no a limit on the number of overshoot intrinsic events between two consequent directional-changes. As the reader can see from figure 1, the minimum size of overshoot is equal to zero which can be observed when the price does not follow the trend and makes a reversal right after a new directional-change point. The pseudo-code of the algorithm is provided in Appendix B.

The breakthrough of the discovered directional-change time is that such a type of measure does not rely on the exogenous evolution of physical time and only endogenous price moves themselves define the stamps at which the measure should be taken. The authors of Guillaume \textit{et al.} (1997) also presented a scaling-law discovered with the help of the new event-based concept, establishing the relationship between the chosen thresholds \( \delta \) and the number of the corresponding trend changes \( N(\delta) \):

\[
N(\delta) = \left( \frac{\delta}{C} \right)^E.
\]  

This scaling law states that in the directional-change intrinsic time the number of directional changes observed in a data sample of the fixed length relates to the size of the chosen threshold as the exponential function.

From figure 1 it is easy to find that due to the non-consistent evolution of financial time series intrinsic time ticks more often during periods of high volatility and ticks less when the market is relatively quiet. In the picture, the last two directional-change events are registered after a weekend (right part of the plot where the exchange rate does not move) simply because during the weekend there were no trades and hence no price moves. Additionally, the directional-change intrinsic time captures only the most valuable information represented by the local maximums and minimums indicating the end of the local trend. All prices between the registered points are considered as noisy and become ignored. The prices, neglected at one threshold can at the same time play the role of tipping points at another threshold. This multi-scale property of the directional-change intrinsic time decreases the signal to noise ratio by providing information only about extreme points of trends at different resolutions defined by the size of the given threshold.

There are various advantages of this event-based approach employed for high-frequency data analysis. Some of them have been successfully used and described in multiple papers. Among them:

(i) The directional-change algorithm was applied to discover numerous scaling laws hidden in the prices from the FX market. The found properties are expected to improve the inferences we make on the price evolution analysing liquid markets (Glattfelder \textit{et al.} 2011);
(ii) Several directional-change thresholds of various sizes initiated at the same time were applied to describe price evolution of the given financial time series at multiple scales and to estimate the unlikeliness of price trajectories. This unlikeliness is later used as the indicator of the current and the future liquidity of the market (Golub et al. 2014);

(iii) At periods of high volatility, the number of directional-changes grows fast. The number stays small when the volatility is close to zero. Thus, the approach can be used as an estimate of volatility of the given time series capturing number and the magnitude of the trend components at multiple scales (Petrov et al. 2018).

A big number of directional changes can be registered if the size of the threshold $\delta$ is significantly smaller than the volatility of the analysed time series. In this case, the average numbers of overshoot intrinsic events observed on the upward and downward local trends will be approximately equal to each other if the overall trend of the time series is equal to zero. Alternatively, if there is a persistent trend, then the price curve will tend to overshoot more when the direction of the local trend at the scale $\delta$ coincides with the direction of the overall trend. Thus, the average numbers of overshoot events upward and downward will not be equal to each other. However, it is possible to modify the original algorithm in such a way that the trend will have no impact on the balance of the number of overshoots. Two distinct thresholds can be used to monitor directional changes independently: $\delta_{up}$ to register events which happen within the upward trend and $\delta_{down}$ for events downward. We include some theoretical analysis in Appendix A where we demonstrate how the trend affects the expected size of overshoot sections and, therefore, the number of directional changes per period of time in case of different trends. As it can be seen from equations A6 and A7, the expected size of the overshoot depends on the trend of the market $\mu$ as well as on its volatility $\sigma$. For example, if the trend is negative then upward overshoots $\omega(\delta_{up})$ are equal to downward overshoots $\omega(\delta_{down})$ only if $\delta_{up} < \delta_{down}$.

Most of the analytical tools developed in finance initially built around the assumption that the market does not exhibit any particular trend. Therefore, some specific trading or risk-management algorithms require precise instruments which could compensate nonzero trend present in the market by adjusting some specific parameters. The directional-change intrinsic time characterised by the aforementioned property of asymmetric thresholds is one of such instruments. However, in the real world the trend cannot be estimated ex-ante so even using the event-based approach it is impossible to say which pair of thresholds $(\delta_{up}, \delta_{down})$ should be chosen to neutralise the given drift specific to the selected period. An efficient solution was proposed in Golub et al. (2017). The authors describe an automated trading algorithm fully exploiting a set of statistical properties accompanying the directional-change intrinsic time. The algorithm tries to efficiently capture the price curve coastline by trading at the moments of alternating trends. To ensure positive returns the equal number of buys and sells should happen within periods of the growing and the falling price sections respectively. In other words, at any moment of time, the algorithm should have precisely selected a pair of upward and downward thresholds. The authors used several levels of accumulated imbalanced inventory as the proxy for the trend. In our work, we are creating a trivial agent-based model where the artificial agents do not keep track of their inventory but each of them, ex-ante, has a unique prespecified pair of non-modified thresholds. Such settings lead to the case when at any moment of time there is an agent whose couple of thresholds capture the current price direction in the best possible way (equal number of upward and downward overshoots). The whole set of agents and their parameters will be described in the following section.

3. Structure of the model

Financial markets can be schematically described as the interplay of two components: a group of financial agents characterised by the diverse range of behavioural patterns and a component responsible for the impact of their aggregated actions on the prices of the exchanged assets.
Two types of sections are typical for each consecutive pair of intrinsic events: directional-change (marked by the solid line) and overshoot (dashed line). The direction of the initial mode is chosen to be downward. The first (left) grey square marks the first directional-change event which occurs when the price moves downward by $\delta$ percents from the local extreme which, for the given mode, coincides with the highest observed price (the first grey circle). The mode alternates (upward) as soon as a new directional-change point becomes registered. After this step, the local extreme coincides with the smallest observed price since the time of the latest directional-change. The next upward event is registered when a positive return of the size $\delta$ happens measured from the local extreme. After this, the mode alternates again and the dissection process continues.

for example, [Delage et al. (2010)]. To keep our model as simple as possible, we stick to this generalised representation. Therefore, the agent-based model has: a set of agents buying and selling fixed volume only at the moment when they register an intrinsic event dependent on the pair of personal thresholds; the volume impact function, a special algorithm generating the next price move (return) as the reaction to the aggregated imbalance of the number of buyers and sellers at the previous price. We do not assume the complete equilibrium of the supply and the demand sides and postulate that the excess volumes can be endlessly bought and sold to counter-parties who have their offers already submitted in the order book.

Real market participants have a diverse set of trading strategies: trading in working days or weekends, technical or fundamental analysis, high-frequency trading or holding long-term positions (see, for example, the survey of US market provided by Cheung and Chinn (2001)). Good agent-based models aimed to mimic real liquid markets should be oriented on the reproducing of similar varied behaviour. We emulate such a complex system by creating a group of intrinsic event agents each of which has a unique set of parameters $\delta_{up}$ and $\delta_{down}$, so there is no a pair of agents with entirely the same settings. This makes the behaviour of individual traders exclusive since the way an agent interpret directional changes of a price curve depends on the size of the given thresholds and their dissimilarity. As a result, various trading activities are reproduced: with small enough thresholds traders register intrinsic events almost at each new price tick (like real high-frequency traders) when with large thresholds traders’ intrinsic time ticks significantly less frequently making their behaviour similar to the practice of long-term investors.

The goal of the agent-based model is to produce a set of returns which then can be converted to a price curve characterised by the same statistical properties found in the FX market. Each
return generated by the price impact function is assumed to be defined in logarithmic terms. This assumption makes it possible to compare distances between prices with the directional-change thresholds directly. In other words, returns between two given prices at steps \( m \) and \( n \) (\( x_n \) and \( x_m \)) are defined as \( r(n, m) \approx \log(x_m) - \log(x_n) = S_m - S_n \) where \( S_m \) and \( S_n \) are sums of returns computed by the volume impact function and accumulated by the steps \( m \) and \( n \). Thus, a new directional-change intrinsic event happens when the absolute distance between the latest price and the local extreme becomes bigger or equal to the delta expressed in absolute terms. This slight simplification significantly facilitates all computations and will be used in the rest of the article.

The whole life cycle of each agent consists of only two actions: opening a long or a short position and waiting for a moment to reverse it. As well as in the real world, the agents do not explore each possible opportunity: they flip their positions with probability \( P_{\text{flip}} \) which makes the performance of the model more realistic. The moment when the agents try to flip the opened positions coincides with the occasion when an intrinsic event occurs in their own intrinsic time determined by the assigned directional-change thresholds \( \delta_{\text{up}} \) and \( \delta_{\text{down}} \). For simplicity, all agents trade identical volume equal to one lot. To flip a position from long to short an agent shorts one unit to close its long position and shorts an additional unit to open a short position. In total two units should be sold. A similar procedure is in place for flipping from a short to a long position, whereas the agent buys two units. Thus, at any iteration \( n \) there are always \( N_{n,\text{long}} \) agents who decided to flip their position from short to long at this step and \( N_{n,\text{short}} \) agents who decided to become short instead of long. The value \( \Delta N_n = N_{n,\text{long}} - N_{n,\text{short}} \) indicates the current excess demand or supply and is used by the model to determine the subsequent price change using the volume impact function. A simplified example of intrinsic events registered by a trader with parameters \( \delta_{\text{up}} = 2, \delta_{\text{down}} = 3 \) is visualized in figure 2.

The whole set of artificial agents should target the diversity of trading strategies present in the real world. This implies that the directional-change thresholds given as parameters for each individual agent have to cover the wide range of moments at which the trading occurs. We selected such a spectrum of values which can register a new intrinsic event as after one elementary price move as well as when the price experiences significant changes measured in large number of ticks. To simplify the model, we represent the whole set of agents as a collection of separate points on a square grid. In this case, values \( \delta_{\text{down}} \) are located along its horizontal side (x-axis) and along the vertical side, there are all included in the model thresholds \( \delta_{\text{up}} \). The distance between two consecutive values on the same axis was selected to be equal to one price tick. This guarantees that any cumulative return represented by the price curve will serve as a new signal for the agent-based model. In figure 3 we demonstrate a part of the grid containing the aforementioned parameters. Each circle represents a couple of thresholds assigned to one unique trader. It is worth mentioning that we experimented with different distributions of thresholds, such as logarithmic distribution along each axis of the grid as well as their radial distribution. In each case, the obtained performance was close to the one offered by the simple squared grid. Nevertheless, the best results were produced by the distribution described in the text. The extent to which the agents cover the diversity of various trading patterns is defined by the geometrical size of the grid: to assure the symmetry of the trading strategies we put \( L \) points horizontally and vertically. Figure 1(a) in Appendix C demonstrates number of trades performed by agents from the whole grid. As expected, agents with the smallest thresholds put in the left bottom corner make the biggest number of trades (high-frequency traders). The right top corner represents rare trading performed by the agents with the biggest thresholds (long-term holders).

The grid can visually be divided into three separate sections. Traders from section I have upward directional-change thresholds larger than the downward ones (\( \delta_{\text{up}} > \delta_{\text{down}} \)). These agents register the equal number of intrinsic events within local trends upward and downwards only when there is a persistent positive trend of the whole time series. If the trend is negative or equal to zero, they tend to exploit more trading opportunities at intrinsic events on the local trend down (descend supporters). Agents from the region II have equal upward and downward thresholds and are called diagonal agents. The most stable conditions for them is the absence of any global trend. In this case,
Figure 2. An example of a price curve dissected by a set of intrinsic events at which the agent with parameters $\delta_{up} = 3$ and $\delta_{down} = 2$ opens a position and makes trades. For simplicity, the chosen thresholds are expressed in the absolute price moves instead of the relative ones. The initial price is equal to 10 and the agent’s initial mode is down. Red arrows mark the distance measured between local extremes and the following directional change points. Green arrows label price moves which lead to the next overshoot intrinsic event. Letters A, B, C and D are put here to tag four intervals of alternating modes. The agent registers its first directional-change intrinsic event as soon as the price goes down from the local extreme by at least 2 points (labelled by a circle). Since this is the first intrinsic event observed by the agent, it opens its first position at this point (step number 2). Then it is waiting for the next event which happens at step 3 after a big price move by four points up from the latest local extreme (coincides with the previous directional change intrinsic event, a grey square). Independently on its previous trading decision, the agent keeps analysing the price curve. At step 5 the price has made an overshoot move measured from the preceding directional-change point the size of which is equal to $\delta_{up}$ (a grey triangle). This point marks the first overshoot intrinsic event. Though the price continues its overshoot move up, it does not go far enough to trigger another overshoot intrinsic event and the next tipping point becomes a directional change at step 8. The next two overshoot intrinsic events happen at steps 9 and 11. The last directional change concludes the example at step 13.

they witness the equal number of intrinsic events within local trends in both directions. Region III marks all agents with upward directional-change thresholds smaller than the downward ones ($\delta_{up} < \delta_{down}$), so their behaviour is the opposite to the traders from the region I (ascent supporters). For each trader from the region I there is a corresponding opposite agent from the region III, therefore, the complete set is fully symmetric and thus imitates the balance of different traders in the real financial world. Later we will show that steady trends periodically observed in financial markets can be simulated by the model via deactivating specific parts of the grid.

The following empirical observation was used to define the volume impact function committed to computing price changes caused by the imbalance between the total demand and supply: the lack of demand motivates the supply side to reduce prices and the lack of supply affects the price rises. The exact shape of the function depends on various factors reliant on selected time horizons where the impact is observed, size of the traded volumes, and types of markets where the trades are performed. Several research works have been done on this topic and different models were proposed. A stable and linear impact function was described by Kyle (1985) and later Huberman and Stanzl (2004) provided the proof that permanent price impact must indeed be linear while the temporary one can be of a more general form. At the same time, more sophisticated non-linear functions were reported (Hasbrouck 1991, Hausman et al. 1992, Kempf and Korn 1999). For our experiments, we decided to choose the model proposed in the relatively recent work Bouchaud (2010). According to Bouchaud, the impact of trading volume is non-linear and one of the best approximations is the square-root function. Therefore, we endow the market response to the agents’ aggregated trades in
Figure 3. A part of the grid representing the collection of the trading agents. Each point corresponds to an agent defined by a set of unique parameters \( \{\delta_{\text{up}}, \delta_{\text{down}}\} \), where \( \delta_{\text{up}} \) and \( \delta_{\text{down}} \) are the size of upward and downward directional-change thresholds correspondingly. By numbers I, II, and III we mark regions with specific properties: in I there are only traders with the upward directional change threshold larger than the downward one \( \delta_{\text{up}} > \delta_{\text{down}} \), the region II contains 'symmetric' agents \( \delta_{\text{up}} = \delta_{\text{down}} \) and III labels all agents with downward thresholds larger than upward ones \( \delta_{\text{up}} < \delta_{\text{down}} \).

The following way:

\[
r_n(\Delta N_n) \equiv S_n - S_{n-1} = \left\lfloor \alpha \cdot \text{sgn}(\Delta N_n) \sqrt{\left|\Delta N_n\right|} \right\rfloor,
\]

where: \( r_n(\Delta N_n) \) is the price change at the step \( n \) also equal to the difference between generated aggregated returns \( S_n \) and \( S_{n-1} \); \( \alpha \) is the parameter limiting the minimum price shift; \( \text{sgn}(.) \) and \( \left\lfloor . \right\rfloor \) are the sign and the floor function correspondingly. There is a situation among all possible scenarios when the number of agents deciding to flip short positions is less than the number of agents deciding to do the opposite just by one agent. This case leads to the imbalanced volume equal to the sum of the volume to close a position and the volume to open an opposite. Therefore, the smallest possible non-zero value of \( \Delta N_n \) is equal to two. The parameter \( \alpha \) was chosen in such a way to guarantee that in that situation the market will respond to this imbalance by changing the price by one unit: \( \alpha = \sqrt{2}/2 \). We use the floor function to simulate only discreet price changes typically observed in real markets. The maximum possible price move \( r_{\text{max}} \) can be observed only when all agents defined in the model decide to flip their positions simultaneously. For this, they all should register intrinsic events at the same moment of time and the opened positions have to be of the same type. Thus, the largest absolute price change is determined by the number of agents on the grid and is directly connected to its length \( L \):

\[
r_{\text{max}} = \alpha L\sqrt{2} = L.
\]
fluctuations enliven further asymmetric activity of the sell and buy sides. Eventually, it becomes portrayed in a sequence of price moves. Similarly, the proposed agent-based model is capable of producing zero difference between the number of all buyers and sellers which entails zero net volume. This volume being put into the equation \( \delta \) does not cause any price change. Without price changes, the agents do not receive any new information and thus do not have a chance to make decisions on further trades. As a result, this puts the next evolution of the price curve on hold.

To reactivate the trading, we added a small random price shift upward or downward with equal probability \( \frac{1}{2} \) which happens whenever the net volume is equal to zero. The size of the random move was chosen to be big enough to trigger a new intrinsic event for agents who have one or both thresholds equal to the smallest price tick: \( \delta = 1 \). This way, these agents receive a new piece of information and try to flip their opened positions thus creating new demand or supply potentially leading to the following price change.

We used the next parameters for all simulations: initial price level \( S_0 = 0 \); minimum price step (a tick) \( \Delta r_{\text{min}} = 1 \); \( \alpha = \sqrt{2}/2 \); smallest and biggest thresholds \( \delta_{\text{min}} = 1 \) and \( \delta_{\text{max}} = 50 \); step between two consecutive thresholds equal to 1; total number of trading agents is 2500. Prices, produced by the model, are aggregated returns computed as responses to all imbalanced volumes that happened in the past. Therefore, the zero initial price simply indicates that no returns happened before the model was initialized. The smallest threshold \( \delta_{\text{min}} = 1 \) guaranties that any elementary price move will trigger a new intrinsic event of agents operating at the minimal scale. The size of the probability to flip position \( P_{\text{flip}} \) coincides with the empirically and theoretically found probability to register a new overshoot event before the directional-change one (Golub et al., 2014):

\[
P_{\text{flip}} = e^{-1} = \mathbb{P}(\omega(\delta) \geq \delta).
\]

\( (4) \)

4. **Benchmarks**

The main goal of this research work is to check whether the agent-based model operating in the directional-change intrinsic time is capable of generating synthetic time series, statistical properties of which are coherent with the ones typical for foreign exchange market. Several benchmarks have been chosen to verify the accuracy of the model. The whole set of tests consists of four traditional methods often used in research with the same intent and a new approach wholly based on the directional-change intrinsic time. We claim the latter is the best way of evaluating agent-based models. Further, we describe all selected benchmarks in details.

4.1. **Traditional benchmarks**

One of the well-known facts about the market microstructures is that price returns in any liquid market do not exhibit significant linear autocorrelation (Arneodo et al., 1996) and ‘in a few minutes’, it can be safely assumed to be equal to zero (Cont et al., 1997). This phenomenon is also formulated as the ‘efficient market hypothesis’: prices instantly and fully reflect all available information making it impossible to build a simple trading strategy based on the ‘statistical arbitrage’ (Basu, 1977).

Only at ultra-short time intervals, when the market as a global multi-agent system is still absorbing a new piece of information, price curves could have slightly correlated historical returns. We selected this statistical property to be among our benchmarks since it is one of the most popular stylized facts regularly accompany liquid markets.

The autocorrelation function of the series of returns \( S \) with mean \( \mu \) and variance \( \sigma \) at the given lag \( \tau \) is defined as

\[
R(\tau) = \frac{\mathbb{E}[(S_t - \mu)(S_{t+\tau} - \mu)]}{\sigma^2}.
\]

\( (5) \)
The second stylized fact is related to the fat-tailed distribution of returns at the frequency higher or equal to one day. This property, also known as excess kurtosis, is thoroughly discussed in the book Mandelbrot and Hudson (2010). The authors point out that despite the full range of theories build on top of the assumption that returns are normally distributed, real financial markets have always been different from Brownian motion and this assumption is a severe flaw of any related financial model.

We measure the excess kurtosis $k$ of the distribution of returns:

$$k = \frac{\langle (r(t, \tau) - \langle r(t, \tau) \rangle)^4 \rangle}{\sigma(\tau)^4} - 3, \quad (6)$$

where $\sigma(\tau)$ is the variance of log-returns $r(t, \tau) = S(t) - S(t - \tau)$ computed with the lag $\tau$. Excess kurtosis $k = 0$ means the absolutely normal distribution and values bigger than 0 indicate deviations from it. Brown and Warner (1985) demonstrated that in the stock market excess kurtosis is usually below 4. Cont (2001) mentioned that for S&P 500 futures the value is around 16, for Dollar/Swiss Franc futures it is 60 and in Gençay et al. (2001) one can find that for USD FX rates it is roughly 30 (through 10 minutes lag intervals). Excess kurtosis is positive when time lag is relatively small and it tends to zero as the lag increases. This fact is called the aggregated normality or aggregational Gaussianity and can be accounted for the ‘mixture of normals’ (Antypas et al. 2013). We select several increasing time lags to demonstrate that the distributing of produces by the agent-based model returns has distinct excess kurtosis dependent on the selected scale. Thus, the aggregated normality is the third benchmark in our set.

The fourth selected benchmark is the scale-invariance of the absolute price change (return) to the period of time when it occurs (Müller et al. 1990, Mantegna and Stanley 1995, Dacorogna et al. 2001). There is no agreement on the origin of this scaling law and various assumptions can be found in the literature (Bouchaud 2001, Farmer et al. 2004, Joulin et al. 2008). Its omnipresence in finance incentivised scientists to apply it for real financial problems such as risk management and volatility modelling (Ghashghaie et al. 1996, Gabaix et al. 2003, Di Matteo 2007). We check whether the returns generated by the directional-change agent-based model are also characterized by this power law and therefore it is our forth benchmark.

It is absolutely essential to be aware of the fact that most of the scaling laws found in financial markets rely on physical time. As we discussed before, it is not efficient in catching extreme events, the most critical information for statistical analysis. Physical time does not adjust its flow to the speed of actions happening at the market. Since agent-based models are neither capable of ‘feeling’ the flow of the physical time (they are a piece of code, after all) it becomes a very complicated question how one can connect the sequence of actions performed by agents to real seconds, days and years. For example, since volatility has a direct impact on the number of observed directional-change intrinsic events (see figure 1), the volatility clustering would simply mean compression and expansion of the inner clock used by an agent-based model. The exact way of how the speed of the clock changes should be algorithmically predetermined. Nevertheless, the most popular approach used to bridge the gap between physical and intrinsic time in agent-based modelling is the assumption that the agents make decisions at equidistant moments of time, for example, every second. In this case, for instance, 20 000 000 steps in intrinsic time would correspond to 231.5 trading days (that is, close to a full trading year of 252 days). To employ the traditional benchmarks, we follow the same principles and postulate that price changes can be observed only in discrete moments of time.

4.2. The ultimate benchmark

Scaling laws are ubiquitous properties of our world and are present in any domain of natural and social phenomena such as physics, biology and social sciences (Andriani and McKelvey 2007). Therefore it is also important to consider these omnipresent multi-scale properties while validating
artificial sets of interacting agents aimed to simulate stylized facts of real financial markets.

The traditional benchmarks mentioned in the previous section all suffer the same drawback: accounting for physical time as the factor of scale. They can be successfully applied for describing real processes and actions existing in the same real space-time world. However, artificial agents have no real ‘feeling’ of time and have only the ‘signal-reply’ logic. Therefore, applying methods and tools where time plays the crucial role for interpretation of the agents’ automated actions in physical time is not fully correct. In the search for a new benchmark which can be ultimately used to verify any agent-based model independently on the extent of discrete time steps, we consulted the work of Glattfelder et al. (2011). There is one, out of 12, newly described scaling laws which are totally agnostic to real physical time: the ‘overshoot scaling law’. It is fully based on the directional-change intrinsic time where only relative price moves dictate sequences of events dismaying time intervals between them. Golub et al. (2014) showed that, in the continuous process with volatility $\sigma$ and zero trend, the probability of overshoot $\omega(\delta; \sigma)$ reaching the length $l$ equals to $\exp\left(-\frac{l}{\delta}\right)$, i.e.

$$\mathbb{P}(\omega(\delta; \sigma) = l) = \exp\left(-\frac{l}{\delta}\right), \quad (7)$$

which reveals the exponential relation between the length of a directional-change threshold $\delta$ and the expected length of the overshoot $\omega(\delta)$. The expected overshoot $\mathbb{E}[\omega(\delta)]$ is equal to the size of the directional change threshold $\delta$:

$$\mathbb{E}[\omega(\delta)] = \delta. \quad (8)$$

Glattfelder et al. (2011) has empirically shown that the average coefficients $E$ and $C$ of the overshoot scaling law across all 13 currency pairs analyzed in the work are $E = 1.04$ and $C = 1.06$. This statistical property is fully agnostic to the volatility of the market and does not depend on physical time. Its application as a benchmark to an agent-based model does not require any additional assumption on the connection between physical and intrinsic time. Therefore, we call this scaling law as the main benchmark of our model.

To validate scaling laws mentioned above we used the same notation proposed in Glattfelder et al. (2011): we build log-log plots where on $x$ and $y$ axis the base and dependent values of scaling laws are put. We assume a linear relationship between the response variable $Y$ (for example, average size of a price return) and the fixed variables $X$ (period of time): $Y = A + BX$, where $A$ and $B$ are unknown parameters to be estimated. Thus $B$ defines the slope of the line on the log-log plot and $A$ is the intersection of $y$ axis. In this way, a scaling law takes the following form:

$$y = \left(\frac{x}{C}\right)^E, \quad (9)$$

where $y = \exp Y$, $x = \exp X$, $E = B$ and $C = \exp(-A/B)$.

5. Results

In this chapter, we highlight the main findings of the research work and review components of the agent-based model affecting properties of generated time series.

Experiments have been performed in two steps: first, we analysed time series generated by the agents from the entire grid operating at the same time; second, we examined the effects induced by the $I$ or $III$ part of the grid being excluded from the model.
5.1. The entire grid

As the first step, we tested the performance of the whole grid of directional-change agents with parameters specified in Section 3. An example of 10 price curves generated by the model with the help of the squared root impact function is presented in figure 4. Every time series does not demonstrate any prevailing direction and consist of various intervals with plateaus and sudden jumps thus mimicking features of the real FX market. The red line represents the average price computed as result of 1000 independent simulations. This line is perfectly horizontal throughout all steps which means that in average the model where all agents participate in the trading does not possess any deterministic trend.

Benchmarks introduced in Section 4 were applied to validate the synthetically generated data sets. Autocorrelation function (ACF) of a time series consisting of 10 000 000 simulated prices is shown in figure 5(a). The maximum negative correlation ($-0.32$) is observed for the lag size equal to one step while the rest of the values is significantly smaller. As expected, the autocorrelation function rapidly decays and becomes indistinguishable from zero after about ten steps. Figure 5(b) shows a slowly decaying autocorrelation function of absolute returns. The bigger the lag step, the lower decline of the ACF.

The absolute price change scaling law is shown in figure 6(a). $C$ and $E$ are the parameters of the scaling law described in section 4 and $R$ is the Pearson product-moment correlation coefficient. It is important to highlight that this scaling law employs physical time as the measure of the distance between prices in the given time series. Since the intrinsic-time models do not directly conduct their trading with the flow of the physical time, several assumptions can be made on the frequency of the generated by the model price moves. The choice of an assumption affects the scaling coefficients in one or another way. Therefore, the fact that the computed correlation is described by a power law function and is represented by a straight line on the log-log plot is enough to confirm the accuracy of the experiment.

Figure 7 shows distributions of returns computed at various lags. There are noticeable fat tails up to lags measured by hundreds of steps and they disappear around the level of 1000 steps. Assuming that a new price move happens every minute, this is in line with empirical results found in real markets (Kullmann et al. 1999). Measured excess kurtosis is equal to 2.73 at 10-steps lag and only 0.06 when the lag rises to 1000-steps. Even though the excess kurtosis rapidly decreases together
Figure 5. (a) Autocorrelation function of the generated time series. Lags are measured in numbers of elementary price moves. (b) Autocorrelation function of absolute returns computed on the same time series.

Figure 6. (a) Average absolute price move as the function of the number of steps (the absolute price move scaling law). (b) Overshoot scaling law build from a time series generated by the agents from the entire grid. Parameters on the plot correspond to the average line. Approximation was done for $\delta > 0.3\%$. The same equation provided in the description of figure 4 was used to transform thresholds from absolute values to relative ones (into percentage terms). Coefficients of the Up line: $C = 1.04, E = 1.05, R = 1.0$; of the Down: $C = 1.03, E = 1.03, R = 1.0$. The plot is based on 20 000 000 steps.

with the growing lag size confirming the empirically observed dependence, the empirical analysis presented in papers mentioned above shows that the excess kurtosis at such small lags is usually much higher ($k = 10$ and even more). However, we found that the size of the grid and thus the diversity of trading agents directly contributes to the measured value: at 10-steps lag the 50 by 50 points grid produces $k = 2.73$, 100 by 100 points $\rightarrow k = 3.46$, 200 by 200 points $\rightarrow k = 6.01$. In figure 8 we present two probability plots for lags of 10 and 1000 steps which once again confirm
aggregated normality.

Finally, we validated the generated time series with the overshoot scaling law by building the log-log plot of the average overshoot length versus the size of the directional-change threshold. Dependence, shown in figure 6(b), appears to be linear with parameters of the scaling law $E = 1.04$ and $C = 1.04$ exceedingly close to the ones observed in the real foreign exchange market (see Section 4.2). In addition to the ordinary experiment described in Glattfelder et al. (2011), we also checked
two supplementary versions of the overshoot scaling law: computed using only overshoots following upward directional changes (red dashed line) and only overshoots following downward directional changes (green dashed line). As it can be seen from the figure there is no noticeable difference between all three lines which confirms results of the theoretical analysis expressed in equations \( A6 \) and \( A7 \) from Appendix A.

Apart from the square-root impact function, linear and logarithmic dependencies have also been tested. Stylized facts of time series generated with the help of these impact functions do not reproduce the same quality revealed by the square-root one. The linear impact induces rapid price fluctuations around the initial level and none of the statistical properties become observed. In turn, time series generated by the logarithmic function are expectantly characterised by the fat-tailed distribution of returns but the overshoot scaling law is still much better imitated by the originally chosen squared-root function. In case of linear or logarithmic impact functions the parameters of the overshoot scaling law are noticeably worse than those based on the square-root one \( (C = 0.86, E = 0.95, R = 0.16 \) and \( C = 1.38, E = 0.99, R = 1.0 \) correspondingly versus \( C = 1.04, E = 1.04, R = 1.0 \).

Markets react to exogenous shocks in multiple ways (Frank and Sanati 2018). Positive and negative news concerning some stock or exchange rate could as permanently change the preceding trend as well as make a short-term disturbance. The latter can become rapidly absorbed by the system or be pronounced in the corresponding price change only some time after the event. Considering successfully replicated by the agent-based model stylized facts we checked how it reacts to small and instant shocks of various amplitudes. Each shock is simulated by a small portion of extra volume on the sell or the buy side added to the net volume created by agents at a specific step. In figure C5 we show the effect the extra volume has on the average aggregated return at each step of the simulation. Obtained results demonstrate permanent impact which is equal to the size of the volume run through the impact equation 2. The average price shift stays constant till the very end of the simulation.

5.2. Asymmetric regions

We showed in the previous section that the whole grid of directional-change agents generates a collection of time series the average trend of which is equal to zero. The reason behind is the symmetry of all active agents: to each agent from the region I with not equal thresholds \( \delta_{up}^I \) and \( \delta_{down}^I \) there is an agent in the region III with symmetrical parameters \( \delta_{up}^{III} \) and \( \delta_{down}^{III} \) such that \( \delta_{up}^I = \delta_{down}^{III} \) and \( \delta_{down}^I = \delta_{up}^{III} \). Remarkable results can be observed when a part of agents from the grid does not participate in the trading: one can direct the average trend of generated price curves upward or downward by tuning the set of used agents. Here we demonstrate a couple of trivial examples where only agents from the region I or the region III have been selected to trade. These settings deviated the average price from the horizontal level correspondingly upward and downward. Results of these two experiments are shown in figures 9 and 10.

The permanent trend observed in both experiments is present due to the imbalance between agents supporting trends of different scales and directions: agents from the region I tend to trade more often when the price is going down when agents from the region III prefer the rising trend. Several factors affect the slope of the average price curve: the total number of agents in the initial grid, the fraction of the grid used to generate a time series, the selected time interval between two consecutive steps. We performed a set of experiments to visualise better the impact of the grid asymmetry on the average price generated by the agents. An extra directional-change agent with the specific pair of thresholds was added to the initial grid of agents. This resulted in 2500 + 1 agents trading simultaneously. A collection of results is presented in figure C3. There we show the distribution of final aggregated returns after 10 000 steps of 1000 independent simulations. Additional agents with symmetric thresholds \( \delta_{up} = \delta_{down} \) have no significant impact on the average aggregated return (parameter \( \mu \) of the Gaussian distribution used for approximations) as well as
on the standard deviation (parameter $\sigma$). Agents with asymmetric thresholds tend to deviate the average trend to the direction which coincides with the direction of their bigger thresholds. The deviation is more pronounced in case of smaller $\delta_{up}$ and $\delta_{down}$. The potential explanation of this effect is that (1) smaller agents similar to high-frequency traders eager react to the trend and (2) they make more trades (see figure 1(a)). A set of overshoots computed on the data generated in each setup is presented in figure C4. Lines related to the upward and downward overshoots deviate from the diagonal line symmetrically.

Grids of different sizes were tested to investigate whether the size of the grid has any impact on the price generation process. Only agents from the $I$ region were involved in the experiment. In figure 1(b) the average aggregated returns of 1000 independent simulations are shown. The grid of bigger size deviates the lines more from the trend-less case. The reason for the phenomenon is in the increasing standard deviation of the net volumes collected after each new price. This has
been also confirmed in another experiment presented in figure C6. There we computed the average positive and negative volume the model creates. As it can be seen from the figure, the bigger the size of the grid, the higher the absolute net volume. This is directly connected to the real world through the fact that global trading systems become more unstable when more market participants are involved in the trading.

A comprehensive analysis of the precise shape and the direction of the trend is a topic for an independent analysis which is beyond of the scope of this research work. Here we would like to note that returns generated by the partially deactivated grid still reproduce stylized facts selected for this project. In figure C2 we include overshoot scaling laws computed using data generated by two separate parts of the grid. The exact value of studied dependencies insignificantly deviates from the ones observed empirically. Therefore, we claim that the given property of the directional-change model can be used to simulate time series symbolised by realistic returns and with the specific predefined trend.

6. Conclusion

A new event-based mechanics underlying the agent-based model was tested in this work. A set of artificial agents mimics the behaviour of real market participants by buying/ selling one lot of the traded asset at the moments when the directional-change intrinsic time ticks. The agents flip their opened positions with probability $P_{\text{flip}}$ at the moments of each new intrinsic event. Two types of intrinsic events are considered: directional changes (moments when the price makes a reversal equal to the size of the directional-change threshold) and overshoot intrinsic events (when price makes an overshoot which is a multiple of the size of the preceding directional change). No other information but a set of prices is needed to identify the moments of the new intrinsic event. Thus, the agents do not rely on time intervals between their trading decisions and consider prices as the only message about the activity in the market.

A wide range of trading strategies observed in the real world was replicated by the model via assigning a unique pair of directional-change thresholds $(\delta_{\text{up}}, \delta_{\text{down}})$ to each simulated trader. The difference between the number of all agents decided to buy or to sell at each new price represents the net volume or imbalance between sellers and buyers. This value was used to estimate the impact of the net volume on the market prices of the traded instrument.

A set of benchmarks was chosen to test the accuracy of the constructed agent-based model. The model is assumed to be successful if it manages to reproduce ‘stylized facts’ discovered in the real financial world, namely: low auto-correlation of returns, fat-tailed distribution of returns, aggregational normality, average price jump and the overshoot scaling laws. A non-linear square-root volume impact function was selected to compute the response of the market on the aggregated trades. The model has successfully passed all five chosen tests. The latter let us make an educated guess that real market participants intentionally or unintentionally make their trades in a very similar way using their own intrinsic time and trend preferences to reverse positions.

Appendix A: Overshoot as the function of trend

As it was shown in the work [Glattfelder et al. 2011], the average length of an overshoot section is approximately equal to the length of the corresponding directional change threshold:

$$\langle \omega(\delta) \rangle \approx \delta. \quad (A1)$$

This dependence was found not only in historical tick data but also in time series generated with help of arithmetic Brownian motion without trend. Nevertheless, analysis of datasets with constant trend revealed that the average length of overshoots do not resemble the length of selected threshold.
and depends on the size of the trend. We select a simple Brownian Motion with increments $dS_t$, trend $\mu$, and volatility $\sigma$ to find the exact form of this dependence:

$$dS_t = S_t - S_{t-1} = \mu dt + \sigma dW_t.$$  \hfill (A2)

Golub et al. (2014) derived the probability of overshoot to reach some fixed value $x$ in case of observing upward overshoot $\omega(\delta_{\text{up}})$ and downward overshoot $\omega(\delta_{\text{down}})$:

$$\mathbb{P}(\omega(\delta_{\text{up}}) \geq x) = \exp \left\{ -\frac{x}{\sigma^2} \cdot \frac{(|\mu| - \mu) + (|\mu| + \mu) \exp \left\{ -\frac{2|\mu|\delta_{\text{up}}}{\sigma^2} \right\}}{1 - \exp \left\{ -\frac{2|\mu|\delta_{\text{up}}}{\sigma^2} \right\}} \right\},$$  \hfill (A3)

$$\mathbb{P}(\omega(\delta_{\text{down}}) \geq x) = \exp \left\{ -\frac{x}{\sigma^2} \cdot \frac{(|\mu| + \mu) + (|\mu| - \mu) \exp \left\{ -\frac{2|\mu|\delta_{\text{down}}}{\sigma^2} \right\}}{1 - \exp \left\{ -\frac{2|\mu|\delta_{\text{down}}}{\sigma^2} \right\}} \right\}. $$  \hfill (A4)

The expected value of the shown probability equation $\mathbb{P}(x) = \mathbb{P}(X \geq x)$ is

$$\mathbb{E}[X] = \int_{0}^{\infty} \mathbb{P}(x) dx,$$  \hfill (A5)

from which it can be found to be equal

$$\mathbb{E}[\omega(\delta_{\text{up}})] = \frac{\sigma^2}{2} \left( 1 - \exp \left\{ -\frac{2|\mu|\delta_{\text{up}}}{\sigma^2} \right\} \right),$$  \hfill (A6)

$$\mathbb{E}[\omega(\delta_{\text{down}})] = \frac{\sigma^2}{2} \left( 1 - \exp \left\{ -\frac{2|\mu|\delta_{\text{down}}}{\sigma^2} \right\} \right).$$  \hfill (A7)

The expected length of the overshoot is just the average of upward and downward expected overshoot, which equals

$$\mathbb{E}[\omega(\delta_{\text{up}}, \delta_{\text{down}})] = \frac{\sigma^2}{2} \left( \frac{1}{(|\mu| - \mu) + (|\mu| + \mu) \exp \left\{ -\frac{2|\mu|\delta_{\text{up}}}{\sigma^2} \right\}} + \frac{1}{(|\mu| + \mu) + (|\mu| - \mu) \exp \left\{ -\frac{2|\mu|\delta_{\text{down}}}{\sigma^2} \right\}} \right).$$  \hfill (A8)

The value depends on four parameters: thresholds $\delta_{\text{up}}$ and $\delta_{\text{down}}$, volatility $\sigma$, and trend $\mu$. In figure A1 we demonstrate the dependence of the overshoot length on various trends when volatility is fixed and, for simplicity, is assumed to be equal to one ($\delta = 1$). It is easy to notice that only in case of zero trend the lengths of upward and downward overshoots coincide with each other, while for any other value not equal to 0 one can observe significant divergence of the curves. The obtained result is quite intuitive: in case of ascending trend price tends to continue its move in the same direction after an upward directional-change event and tends to make a reversal after a downward event. This asymmetry lead to the case when upward overshoots are in average appreciably longer than the downward ones.

This observation suggests that for each steady trend $\mu$ and volatility $\sigma$ there are such thresholds $\delta_{\text{up}}$ and $\delta_{\text{down}}$ that the total number of directional-change events registered in the given time series would
is constant. This property was directly used in our agent-based model when we designed a set of trading agents with the full range of parameters defining their behaviour (Section 3).

Appendix B: Dissection Algorithm

By $S_{tick}$ we mark the latest observed price, by $S_{ext}$ the local extreme, mode is the current mode of the alternating trend which can be equal either up or down, $\delta_{up}$ and $\delta_{down}$ are upward and downward thresholds respectively, $S_{IE}$ is the price at which the latest intrinsic event was observed. The algorithm returns 1 and −1 when the price curve hits levels of upward and downward directional-change events correspondingly. 2 and −2 will be returned in case of overshot intrinsic events registered on ascending or descending trends.
Algorithm 1 Intrinsic Event

1: if first tick then
2: \( S_{ext} \leftarrow S_{tick} \)
3: \( S_{IE} \leftarrow S_{tick} \)
4: \( \text{return } 0 \)
5: else if mode is up then
6: \( \text{if } S_{tick} - S_{ext} \geq \delta_{up} \text{ then} \)
7: \( \text{mode} \leftarrow \text{down} \)
8: \( S_{ext} \leftarrow S_{tick} \)
9: \( S_{IE} \leftarrow S_{tick} \)
10: \( \text{return } 1 \)
11: else if \( S_{tick} < S_{ext} \) then
12: \( S_{ext} \leftarrow S_{tick} \)
13: \( \text{if } S_{IE} - S_{ext} \geq \delta_{down} \text{ then} \)
14: \( S_{IE} \leftarrow S_{tick} \)
15: \( \text{return } -2 \)
16: else
17: \( \text{return } 0 \)
18: else if mode is down then
19: \( \text{if } S_{ext} - S_{tick} \geq \delta_{down} \text{ then} \)
20: \( \text{mode} \leftarrow \text{up} \)
21: \( S_{ext} \leftarrow S_{tick} \)
22: \( S_{IE} \leftarrow S_{tick} \)
23: \( \text{return } -1 \)
24: else if \( S_{tick} > S_{ext} \) then
25: \( S_{ext} \leftarrow S_{tick} \)
26: \( \text{if } S_{ext} - S_{IE} \geq \delta_{up} \text{ then} \)
27: \( S_{IE} \leftarrow S_{tick} \)
28: \( \text{return } 2 \)
29: else
30: \( \text{return } 0 \)

Appendix C: Additional experiments
Figure C1. (a) Average number of trades done by agents with parameters ($\delta_{\text{up}}$, $\delta_{\text{down}}$) within 10 000-steps simulation. (b) Impact of the grid size on the average return of 1000 independent simulation when only agents from the region $I$ are active. The smallest grid is 10 by 10 points. The biggest is 130 by 130. Used increment is 10.

Figure C2. Overshoot scaling laws computed on the time series generated by the agents from (a) the $I$ and (b) from the $III$ regions only. Red dashed line (Up) corresponds to overshoots computed after directional changes upward. Blue dashed line (Down) corresponds to overshoots measured after directional-change downward. The scaling parameters $C$, $R$, and $R$ are shown for the line representing the average overshoot lengths (blue dashed line).
Figure C3. Impact of the additional agent. Here we investigate the impact of an additional agent trading simultaneously with all agents from the initial squared grid. Parameters of all supplementary agents are put on top of subplots. The number of steps in each simulation is 10 000. Normal distribution approximates the distribution of final prices. Blue mark labels zero level of the final price which corresponds to the case with no trend. Red mark stands for the centre of the distribution. Each picture is built on top of 1000 independent simulations and centred on the average aggregated return.
Figure C4. Overshoot scaling laws computed on time series generated by the whole grid of agents plus one extra.
The parameters of the additional agent are shown on top of each subplot. Green line corresponds to overshoots computed on the downward trend and red line to the overshoots captured on the upward trend. The black line is the average of both lines. The parameters $C$ and $E$ correspond to the average line.
Figure C5. Here we demonstrate the impact the additional volume has on the average price computed as results of 1000 independent simulations. The whole initial grid has been trading without any changes until the step 2000. At this step the net volume generated by the model was increased by the following values: $\pm 1, \pm 4, \pm 9, \pm 16, \pm 25, \pm 36, \pm 49$. Impact of additional volume on the average computed using 1000 independent simulation. The top and the bottom lines are results of changing the net volume at step 2000 by 49 and $-49$ units correspondingly.

Figure C6. Impact of the grid size on the average net volume. Blue curve corresponds to the average excess of sell volume. Red line corresponds to the average excess of the buy volume. Average total net volume is equal to zero.
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Agent-Based Model Replicating Position Ratio at Forex Market

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1 Position information

This work is the extension of the original paper Petrov et al. (2018) where an agent-based model replicating statistical properties of the Forex market is presented. The extension contributes to the overall project by demonstrating how artificial agents operating in directional-change intrinsic time are able to reproduce similar net position ratio observed at the currency trading platform OANDA1.

Net position is the percentage difference of all currently opened short and long positions. For example, the value equal to 20% means that the fraction of long and short positions at the given market are 60% and 40% respectively. The evolution of the position ratio is typically represented by a graph where its historical values are shown for the selected currency pair and over a given time period.

The net position ratio has substantial value when the risks associated with the traders’ activity are discussed. Substantial net values can be considered as a sign of the upcoming market destabilisation (Wang, 2002). Positive as well as negative correlations between the ratio and the volatility of market prices have been observed and described in Chang et al. (2000). The study of the origin and the evolution of the net ratio is essential to manage the potential risks associated with traders’ response to various price changes.

In this extension of the original agent-based model, we pursue the goal of comparing historical position ratios registered at a real trading venue with the net positions formed by artificial agents operating in directional-change intrinsic time.

The net position ratio reflects the reaction of traders to the observed price moves. The initial structure and parameters of the agent-based model have been insignificantly modified in order to change the model from the price-generating to the price-reacting one. In other words, the price impact function has been removed and the real historical tick-by-tick data has been used as the only input parameter. Another change concerns the

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way how the price evolution is interpreted by each agent: decisions on whether to enter a position or not is strictly connected to the sign of the current trend. For example, if an agent observes a downward intrinsic event and if he has opened short position than he can flip it with the probability $P_{\text{flip}}$. Otherwise, the event is ignored. Overall, modified logic is a bit more sophisticated in contrast to the previous price interpreting and decision-making activity.

Examples of position ratios formed by agents as the reaction to price changes are presented in Figures 1a-1d. It is easy to notice how the curve generated by the agent-based model (red line) closely resembles values formed by real market participants (green line). Curves, presented on the pictures, were obtained using the grid of 20 by 20 agents where the smallest and the biggest directional-change thresholds are equal to 0.01% and 2% correspondingly. The selected probability to flip a position is 0.1. It is important to note, that results of performed experiments are noticeably susceptible to the parameters of the model. The best fit of all four real net ratio curve was used as the proxy for the grid size and the probability values.

The high resemblance of real and simulated net curves confirms that real market participants interpret price returns and react to their changes in the way similar to the behaviour of artificial agents operating in directional-change intrinsic time. That is, there are various categories of traders characterised by trading strategies dependent on specific scales and intensities of price moves. This knowledge changes the way scientists and econometricians used to interpret the evolution of price curves: periodicity agnostic directional-change intrinsic time instead of the classical calendar time. This change lays the ground for further research where new scale and time-invariant risk-management tools will be studied.

References


Figure 1: Historical net position ratio versus the position ratio of directional-change intrinsic time agents. The agents flip their position with probability 0.1 at each new intrinsic events if the sign of the trend coincides with the sign of their positions.
Instantaneous Volatility Seasonality of Forex and Bitcoin Exchange Rates in Directional-Change Intrinsic Time

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We uncover weekly instantaneous volatility seasonality patterns of three Forex and Bitcoin exchange rates using a novel volatility measure that does not depend on the moment when the algorithm is initialised and is applicable to unfiltered tick-by-tick data. The measure adapts sequences of drawdowns and drawups as indicators of market activity instead of classical log-returns computed through equal intervals. The sequences are re-expressed in terms of directional-change intrinsic time where ticks happen only when price curve changes the direction of its trend. We use Monte Carlo simulations to confirm the analytical expressions connecting characteristics of the intrinsic time and the properties of analysed prices. We demonstrate the long memory of instantaneous volatility by reconstructing the volatility pattern in terms of non-equal time intervals through which the aggregated price activity is constant. Provided volatility estimation method can be adapted as a universal multiscale risk-management tool not restricted by the continuity of physical time and the type of analysed data.

Keywords: Instantaneous volatility; Intrinsic time; Scaling laws; Forex; Bitcoin; Risk management

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1. Introduction

One of the most well-known risk-factors in Finance is the probability of so-called drawups and drawdowns, that is, of price drops and price rises between the running price maxima and running minima respectively. Numerous research works have focused on the analysis of the size, periodicity and the time of recovery associated with drawups and drawdowns in traditional markets. The joint Laplace transform was utilised by Taylor (1975) for deriving the waiting time $\tau_a$ for a drifted Brownian motion. The joint probability of observing a drawup of a given size after a drawdown, during a given term, was analysed as a homogeneous diffusion process (Pospisil et al. 2009). Zhang (2015) derived the same probability in the context of exponential time horizons (the horizons are exponentially distributed random variables) and described the law of occupation times for both

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drawup and drawdown processes. These and others theoretical findings were successfully applied to real financial problems briefly discussed in Appendix [A].

The cornerstone element of all research works on drawdowns is the first passage time $\tau_a$. This estimate, associated with market crashes of various scales, is especially useful of handling the dynamics of high-frequency markets. The reason for it is the fact that in reality, extreme price jumps occur more often than it should happen when the distribution of returns coincides with the normal one. Fat-tails effect were discovered in the stock market [Jondeau and Rockinger 2003, Rachev et al. 2005], in the Forex [Dacorogna et al. 2001, Cotter 2005] as well as in Bitcoin prices [Liu et al. 2017]. Therefore, the efficient set of forecasting techniques aimed at identifying appropriate conditions for the future market crashes should inevitably be supplied by risk-management tools managing sequence of drawdowns and drawups.

Existing literature on risk-management techniques primarily relies on physical time as a measure of the length and periodicity of financial events. In other words, the existence of a universal clock dictating the evolution of the market is assumed. This assumption results in the stochastic volatility of historical returns computed at different scales [Müller et al. 1997]. More robust techniques which are beyond the limits of physical time applied for studying financial activity are needed to handle this stochasticity. One of the methods capable of doing so is the concept of directional-change intrinsic time [Guillaume et al. 1997]. This is an event-based framework where the activity of market prices dictates the speed of the transition between different states. The universal physical clock is replaced with an intrinsic one. The method dissects a price curve into sections characterised by alternating trends of arbitrary defined size. The concept is closely related to the evolution of drawdowns and drawups. One can interpret their alternating sequence as a collection of directional changes following each other. Physical time does not play any role in the dissection procedure. More details on the directional-change intrinsic time can be found in Appendix [B].

One of the most prominent and still insufficiently studied objects is the rapidly evolving world of cryptocurrencies. The most famous representative of that world is its first successful pioneer Bitcoin. Bitcoin was created in 2008 as an alternative to the classical financial system and as the ‘peer-to-peer version of electronic cash’ [Nakamoto 2008]. Bitcoin and its underlying technology blockchain swiftly gained attention from the technologically savvy community and media and soon became one of the most debatable topics at all levels of the modern society. Over a thousand alternative cryptocurrencies, based on the similar concept, emerged since the time Bitcoin was invented. Some part of them became accessible for trading at various electronic venues also known as crypto-exchanges. In contrast to the traditional FX market, trades of cryptocurrencies happen 24 hours a day and seven days per week. Additionally, there are not many big centralised financial institutions capable to directly influence the state of the system. This happens due to the limited acceptance of this new financial domain by international organisations with the access to sizable funds. These specific facts are the reasons why the seasonality patterns prevalent in the world of cryptocurrencies could be incomparable with the ones typical for the FX market.

Bitcoin’s trends drastic changes and their persistence do not only indicate the aggregated expectations and actual actions of all market participants but also reveal its sensitivity to exogenous stress factors. The latter is known as liquidity and is directly related to the scale of market crashes as well as to the periods of recovery [Gennotte and Leland 1990]. Asset managers and options seller try to foresee future risks associated with the trend changes by employing a diverse range of risk management tools capable for estimating the probability of price disturbances.

In this research work, we investigate the connection between the observed number of alternating drawdowns and drawups (directional-changes) and the instantaneous volatility. Non-parametric estimation of instantaneous volatility is still a relatively new topic which, to the extent of our knowledge, has not been studied before from the point of view of the directional-change intrinsic

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1 A growing list of records containing information on the ownership of all existing Bitcoins.
2 Information on all cryptocurrencies and trading venues can be found at the web-resource coinmarketcap.com.
3 At the moment of writing the paper Wall Street and other big financial hubs are considering trading cryptocurrencies.
time. Obtained analytical expressions are employed to reveal the seasonality structure of instantaneous volatility typical for exchange rates from the Forex market. The latter is compared to the seasonality patterns of Bitcoin market. Described experiments contribute to the collection of existing literature on the seasonality properties of high-frequency markets and especially Bitcoin a brief description of which can be found in Appendix C.

The outline of the remaining paper is as follows. Section 2 describes the data used in the experiments and Section 3 outlines the way in which the number of directional changes is connected to the instantaneous volatility. In Section 4 we present all results obtained by the traditional as well as by the novel volatility measurement techniques and also describe the theta time applied to remove the seasonality pattern. Section 5 concludes the main body of the paper and proposes the potential use of the developed technique. Appendix A provides a brief overview of research works on the properties of drawdowns and drawups. Appendix B gives detailed reasoning of the need in the directional-change intrinsic time and contains a set of literature where the concept was successfully applied. In Appendix C the main findings of research works on seasonality patterns of cryptocurrencies are provided. Appendix D includes additional results of the volatility computed by the ‘natural’ estimator and Appendix E contains results of the Monte Carlo simulation confirming the accuracy of the model.

The fully functional code used in the project can be downloaded from the author’s GitHub repository.

2. Data

Three FX exchange rates were used in the work: EUR/USD, EUR/JPY, and EUR/GBP. The covered time interval is from January 2011 to January 2016 and includes 109 069 357, 134 737 397, and 88 704 676 ticks correspondingly. The source of the data is JForex trading platform developed by the Swiss bank and marketplace Ducascopy which provide various types of the market data in the highest resolution.

Bitcoin price changes observed at the Kraken crypto-exchange were downloaded from the Bitco-incharts online platform supplying financial and technical data related to Bitcoin network. The studied time interval is from January 2014 to April 2018 and includes 4 778 429 ticks.

2.1. Inner price

Any collection of historical prices typically assumes two values: the best bid (buy) and the best ask (sell) prices. Non-zero difference between them, called spread, indicates the level of markets’ liquidity (Bessebinder 1994, Menkhoff et al. 2012). It also has direct connection to realised volatility (Bollerslev and Melvin 1994), and is an indicator of the transaction cost (Hartmann 1999). Another role of the spread is to show the extent of uncertainty the market has on the fair price of the traded asset. The level of the uncertainty constantly changes over time together with the size of the spread. This fact does not allow us to employ only bid or ask prices to study properties of the whole market without missing some part of the information. The average of these two values (mid-price) is also not the best alternative since it does not keep the knowledge on the size of the current spread. Therefore, an alternative measure should be chosen to apply the directional-change algorithm to real data. The concept of inner price was selected to resolve the issue. Inner price can temporarily be equal to the bid or the ask price depending on the direction of the current trend. According to the directional-change algorithm, one should wait for the price increase by $\delta$ percents from the local minimum to register a new directional change if the current trend is down. In this

1https://github.com/VladUZH/VIPetrov
2https://www.dukascopy.com/swiss/english/forex/jforex/
3http://api.bitcoincharts.com/v1/csv/
case, the value of the local extreme coincides with the best price on the offer side of the order book, that is, the ask price. A new intrinsic event will tick only when the distance between the latest bid price and the local extreme reaches the size of the chosen directional-change threshold \( \delta \). Alternatively, the local extreme takes the value of the best bid price and the distance is measured between the newest ask and the extreme if the current mode is upward.

3. Methods

Theoretical researchers mostly rely on the Brownian motion as the proxy for price returns of real markets (the most famous example is the work of [Black and Scholes (1973)]). The motive behind this is the analogy between historical price moves and changing coordinates of an ensemble of molecules in thermodynamics. In the classical work [Osborne (1959)] the author shows that the steady-state distribution of log-returns in the stock market is the probability distribution for a particle in Brownian motion. It is important to emphasise that at the time of the publication (1959) the structure and the dynamic of the market was very different from the ones typical for our modern digital world. Instead of stone-made trading floors where all deals happened more than a half a century ago, the present-day trading has almost completely moved to the online space (see [Harris (2003)] for the historical endeavour on the evolution of trading and exchanges). In this space, a signal can easily propagate through borders with the speed of light and the majority of trades happen in an automated way. The evolution of price returns acquired specific dynamic characterised by a set of stylized facts. These facts, dependent on the selected for the observation time scale, reveal deviations of real returns from the classical Brownian motion (see [Cont (2001)] for the set of stylized empirical facts). Nevertheless, in our work, the choice of Brownian motion is justified by two reasons. First, the directional-change concept is agnostic to the flow of physical time and is capable of revealing even weak signals hidden in a collection of prices at multiple scales. Second, the divergence of empirical results from the properties of the selected model helps to understand the features of the real world markets better. Therefore, we model the set of prices \( \{S_t : t \geq 0\} \) as an arithmetic Brownian motion with trend \( \mu \) and volatility \( \sigma \):

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dB_t. \tag{1}
\]

In terms of the directional-change intrinsic time framework, \( T_{up}(\delta_{up}) \) denotes the time for an upward directional change of the size \( \delta_{up} > 0 \) to unfold. In other words, it is the time interval which passes until the price increases by \( \delta_{up} \) percents from the local minimum \( m_t \). Technically:

\[
T_{up}(\delta_{up}) = \inf\{t > 0 : S_t - m_t \geq \delta_{up}\}, \tag{2}
\]

where

\[
m_t := \inf_{\epsilon \in [0,t]} S_\epsilon. \tag{3}
\]

Similarly, \( T_{down}(\delta_{down}) \) is the time of a downward directional change of the size \( \delta_{down} > 0 \):

\[
T_{down}(\delta_{down}) = \inf\{t > 0 : M_t - S_t \geq \delta_{down}\}, \tag{4}
\]

where

\[
M_t := \sup_{\epsilon \in [0,t]} P_\epsilon. \tag{5}
\]

Both of these equations are also known in the literature as waiting times of drawups and drawdowns.
definition of which is provided by equation A.3. In Landriault et al. (2015) it is shown that expected times of a drawup \( \delta_{up} \) and a drawdown \( \delta_{down} \) depend on the volatility and the trend of the drifted Brownian motion and can be mathematically expressed as

\[
\mathbb{E}[T_{up}(\delta_{up})] = \frac{e^{-\frac{2\mu}{\sigma^2}\delta_{up}} + 2\mu \delta_{up} - 1}{2\mu^2},
\]

and

\[
\mathbb{E}[T_{down}(\delta_{down})] = \frac{e^{\frac{2\mu}{\sigma^2}\delta_{down}} - 2\mu \delta_{down} - 1}{2\mu^2}.
\]

Using the Taylor expansion \( e^{\pm \frac{2\mu}{\sigma^2}\delta} = 1 \pm \frac{2\mu}{\sigma^2}\delta + \frac{(2\mu/\sigma^2)^2}{2!} + \mathcal{O}(\mu^3) \) and letting \( \mu \to 0 \), one can recover that in the case with no trend the equation simplifies to

\[
\mathbb{E}[T_{up}(\delta)] = \mathbb{E}[T_{down}(\delta)] = \frac{\delta^2}{\sigma^2}.
\]

These equations establish a scaling law dependence between waiting times of a directional change, instantaneous volatility, and the size of the directional-change threshold. In the analysis of Glatzfelder et al. (2011) it was empirically found that in the FX market the average expected time is proportional to the size of the directional change threshold \( \delta \) used to identify alternating trends raised to the power of two:

\[
\langle T(\delta) \rangle \sim \delta^2,
\]

which confirms the assumption that the evolution of real prices has similar properties to the random walk.

Let \( N(\delta_{down}; \sigma, \mu, [0, T]) \) denote the number of drawdowns of the size \( \delta_{down} \) observed within the time interval \([0, T]\) in Brownian motion with parameters \( \mu \) and \( \sigma \). Since the sequence \( T_{down}(\delta_{down}) \), \( T_{down}(\delta_{down})^2 \), \ldots is the sequence of non-negative, independent and identically distributed random variables, the sequence \( \{\psi_n; n \in \mathbb{N}\} \) where \( \psi_n = T_{down}(\delta_{down})^1 + \ldots + T_{down}(\delta_{down})^n \) is the renewal point process. Thus, \( N(\delta_{down}; \sigma, \mu, [0, T]) \) can be considered as the renewal counting process and its values can be found using the waiting time equation 7 and applying the Theorem 6.1.1 of Rolski et al. (2009) (Landriault et al. 2015):

\[
\lim_{T \to \infty} N(\delta_{down}; \sigma, \mu, [0, T])^{-1} = \mathbb{E}[T_{down}(\delta_{down})]^{-1} = \frac{2\mu^2}{e^{\frac{2\mu}{\sigma^2}\delta_{down}} - 2\mu \delta_{down} - 1}.
\]

Correspondingly, the expected number of drawups \( N(\delta_{up}; \sigma, \mu, [0, T]) \) takes the form

\[
\lim_{T \to \infty} N(\delta_{up}; \sigma, \mu, [0, T])^{-1} = \mathbb{E}[T_{up}(\delta_{up})]^{-1} = \frac{2\mu^2}{e^{-\frac{2\mu}{\sigma^2}\delta_{up}} + 2\mu \delta_{up} - 1}.
\]

Equations 10 and 11 combined together, give the estimate of the number of directional changes consequently following each other:

\[
\mathbb{E}[N(\delta_{up}, \delta_{down}; \mu, [0, T])] = \frac{2T \frac{2\mu^2}{\sigma^2}}{e^{-\frac{2\mu}{\sigma^2}\delta_{up}} + e^{\frac{2\mu}{\sigma^2}\delta_{down}} + 2\mu (\delta_{up} - \delta_{down}) - 2}.
\]
In the trend-less case the expression is simplified to the following form:

\[
\mathbb{E}[N(\delta_{\text{up}}, \delta_{\text{down}}; \sigma, [0,T])] = \frac{2T \sigma^2}{\delta_{\text{up}}^2 + \delta_{\text{down}}^2}.
\]  

(13)

These theoretical dependencies between the number of directional changes and the properties of the analyzed time series are equivalent to the empirical observations found in Guillaume et al. (1997). There the authors mention that \(N(\delta) \sim \delta^{-2}\) (for \(\delta = \delta_{\text{up}} = \delta_{\text{down}}\)).

A Monte Carlo statistical test was performed to numerically verify the accuracy of the obtained results expressed in equations [6, 7] and [12]. Results of the experiment are provided in Table [D1]. We selected only positive values of \(\mu\) since the equations are symmetrical to the direction of the trend. All experiments exhibit high similarity of both empirical and theoretical values.

The meaning behind the provided equations is that the absolute size and the ratio of directional-change thresholds used to dissect a price curve in a sequence of upward and downward trends affect the frequency and the total number of events within a given time interval. In figure [1] we demonstrate three heatmaps where each point corresponds to the number of directional changes registered with a pair of thresholds \(\{\delta_{\text{up}}, \delta_{\text{down}}\}\). Each heatmap represents the results of a Monte Carlo simulation where Brownian motions with different parameters are applied. From equations [10] and [11] it follows that the combination \(\gamma = \frac{\mu}{\sigma^2}\) is the crucial factor affecting the expected number of intrinsic events. The expression \(\gamma\) often appears in the intrinsic time framework and is known in the insurance industry as ‘adjustment coefficient’ or ‘the Lundberg exponent’ (Asmussen and Albrecher 2010). It finds its application in the ruin theory dating back to 1909 (Lundberg 1909). It is also described as the optimal information theoretical betting size called Kelly Criterion (Kelly Jr 2011). In this work, we check three different scenarios affecting the size of \(\gamma\): \(\frac{\mu}{\sigma^2} = 0\) (figure [1(a)], \(\frac{\mu}{\sigma^2} \ll 0\) (figure [1(b)], and \(\frac{\mu}{\sigma^2} \gg 0\) (figure [1(c)]).

Panel [1(a)] in figure [1] corresponds to the set of experiments where the trend of the Brownian Motion is equal to zero. From equation [13] it follows that in such conditions the value \(\mathbb{E}[N(\delta_{\text{up}}, \delta_{\text{down}}; \sigma, [0,T])]\) should be constant along circular contours \(\delta_{\text{up}}^2 + \delta_{\text{down}}^2 = \delta^2\) for \(\delta > 0\).

The provided picture confirms the noted dependence. In insets [1(b)] and [1(c)] of figure [1] it is shown that when the ‘adjustment coefficient’ \(\gamma = \frac{\mu}{\sigma^2}\) is significantly smaller or bigger than zero, the circular contours transform into ellipses. This phenomenon can be interpreted in the following way: if \(\mathbb{E}[N(\delta_{\text{up}} = \delta_{\text{down}}; \gamma = 0, [0,T])]\) is the expected number of directional changes registered in the drift-less time series of given length and fixed \(\sigma\) then for any \(\gamma\) greater or smaller then zero there...
is always such a couple of non equal thresholds \( \{\delta_{\text{up}}, \delta_{\text{down}} | \delta_{\text{up}} \neq \delta_{\text{down}} \} \) that

\[
E[N(\delta_{\text{up}}, \delta_{\text{down}} | \delta_{\text{up}} \neq \delta_{\text{down}}, \gamma \neq 0, [0, T])] = E[N(\delta_{\text{up}}, \delta_{\text{down}} | \delta_{\text{up}} = \delta_{\text{down}}, \gamma = 0, [0, T])].
\] (14)

The property is essential for risk management techniques constructed on top of directional-change intrinsic time: any process characterized by certain degree of persistent trend could be treated as the one without the trend by tuning the size and the ratio of selected directional-change thresholds. An example of real application of this fact is provided in Golub et al. (2017) where the authors employ it to design optimization inventory control function sensitive to the trend changes.

The clear understanding of the way how volatility changes over time is particularly important for risk management and inventory control problems. As discussed in Section 4, classical volatility estimation methods, also called ‘natural’ estimators (Cho and Frees 1988), primarily rely on physical time as the core reference for historical returns. Stochastic volatility accounted for this fact became a cornerstone for multiple research works (for example, Andersen and Lund (1997), Barndorff-Nielsen and Shephard (2002), Ai et al. (2007) and many others). Values, computed by ‘natural’ estimators, dominantly correspond to the integrated volatility of the studied process. Alternative estimators, designed to reveal the size of the instantaneous volatility, mostly based on Fourier analysis and require extensive computations (see Chapter 3 in Mancino et al. (2017)). The directional-change concept is by design agnostic to the speed of the time flow and it automatically adapts to the periods of changing activity. This property of the directional-change intrinsic time together with analytical equations 12 and 13 bring the idea of a new instantaneous volatility estimator devoid of the equidistant timestamps shortcomings. From equation 13 it follows that for a trend-less time series the instantaneous volatility can be estimated by counting the number of directional-changes within the time interval \([0, T]\):

\[
\sigma_{\text{DC}} = \delta \sqrt{\frac{N(\delta)}{T}}.
\] (15)

We put the superscript \( \text{DC} \) to distinguish the volatility computed through the directional-change intrinsic time from the traditional estimators. The latter we will mark by \( \sigma_{\text{trad}} \).

Equation 15 solely computes the volatility part of the process in case of a random walk. Therefore, it can be classified as the true estimator of the instantaneous volatility. In the current work, we apply equation 15 to study changing dynamic of financial time series throughout one week to reveal its seasonality pattern.

### 4. Results

#### 4.1. Number of directional changes

Historical returns from real markets have properties similar to the Brownian motion used to derive equations 12 and 13 connecting the expected number of directional changes with parameters of the underlying process. Similar counters shown in figure 1 should be found in heatmaps depicting the number of directional changes empirically registered in real data if the assumption of the normal distribution of returns is true. EUR/USD and BTC/USD exchange rates were taken to verify the statement by replicating the same experiment done with the Brownian motion before. A collection of 40 directional-change thresholds ranging from 0.1% to 4.1% defines the scale of the heatmap grid. Results of the experiments are presented in figure 2. Colour schemes, used for both plots, have different scales due to the difference in EUR/USD and BTC/USD volatility. Yellow lines

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1The work Cho and Frees (1988) is particularly interesting due to the analysis the authors did to compare volatilities computed by ‘natural’ and ‘temporal’ estimators. The latter employs time intervals measured between consequent and alternating price moves of fixed relative size and thus is very close to the approach presented in the current paper.
indicate areas of the equal number of directional changes corresponding to $\delta = \{1.15\%, 2.8\%\}$ in case of EUR/USD and to $\delta = \{1.4\%, 3.0\%\}$ in case of BTC/USD. Curves in figure 2(a) have almost circular shape and are only slightly shifted towards the bigger $\delta_{down}$ values. This shift is present due to the downward trend experienced by the exchange rate from 2011 to 2016 (from $1.4$ to $1.1$ per Euro). EUR/USD time series exhibited relative stability with no noticeable regime switches apart from this constant trend. BTC/USD exchange rate was much more unstable concerning the dynamic of EUR/USD. The price was relatively steady and did not demonstrate any permanent non-zero trend from 2014 to 2017. The price of Bitcoin grew with accelerating pace by more than 20 times in the second half of 2017 and then lost nearly 70% of its value at the beginning of 2018. These significant trend changes are illustrated in figure 2(b) by yellow counters with a notable deviation from the circular shape. This shape can be decomposed into two parts of independent ellipses similar to the ones observed for Brownian motion with non-zero ‘adjustment coefficient’ $\gamma$ (Figures 1(b) and 1(c)). The price roller-coaster caused considerable disparity of the number of registered directional changes by any pair of thresholds before and after the pick at the end of 2017.

4.2. realised versus instantaneous volatility

In the second experiment, we compared the annualised volatility computed by the traditional method (equation 16) and the volatility based on the observed number of directional changes (equation 15). Returns $R_t$ are computed as logarithms of the ratio of consecutive prices $S_t$ and $S_{t-1}$ in the traditional technique. The length of the sample $n$ depends on the selected time interval between two measures of the returns. The whole sample is used to find the standard deviation of the time series also known as realised volatility $\sigma_{trad}$:

$$ R_t = \ln \left( \frac{S_t}{S_{t-1}} \right), \quad R_{avg} = \frac{\sum_{t=1}^{n} R_t}{n}, \quad \sigma_{trad} = \sqrt{\frac{\sum_{t=1}^{n} (R_t - R_{avg})^2}{n - 1}}. $$

It had a minimum at $230$ per Bitcoin, temporary maximum at $20\,000$, and then a drop to $6\,000$. 
The directional-change method does not define the number of observations ex-ante in contrast to the traditional approach where the length of a sample depends on the preselected time span between consecutive returns. According to equation \( \alpha \) the size of the directional-change threshold \( \delta \) determines the expected range of the number of prices used to calculate the instantaneous volatility. This flexibility of the intrinsic time makes it possible to use the data of the highest frequency: tick-by-tick prices.

Parameters of tools used to estimate volatility can affect the results of the experiment \cite{Müller et al. 1997}. Therefore, four increasing time intervals \( \Delta t_k \) where \( k = \{1, 2, 3, 4\} \) were selected to define the distance between each pair of consecutive prices \( S_t \) and \( S_{t-1} \) used for the ‘natural estimator’: \( \Delta t_1 = 1 \) minute, \( \Delta t_2 = 10 \) minutes, \( \Delta t_3 = 1 \) hour, and \( \Delta t_4 = 1 \) day. The set of thresholds employed to investigate the directional-change approach can also be arbitrarily chosen. However, we initially targeted the goal to compare the results of both experiments. For this reason, we used the number of timestamps corresponding to each time interval \( \Delta t_k \) as the target for the number of directional changes registered in the same data set. That is, the collection of four thresholds \( \delta_k \) was selected in such a way that in the given time series the number of directional changes will be approximately equal to the number of time intervals \( n_k \) of the length \( \Delta t_k \). We utilised one of the scaling properties described in \cite{Glattfelder et al. 2011} to find the precise thresholds size. The scaling property has the name ‘Time of total-move’ scaling law \cite{C 2001} where the total-move is composed of a directional-change (DC) and an overshoot (OS) parts. The law connects the size of the threshold \( \delta \) with the waiting time \( T_{TM}(\delta) \) between two consecutive intrinsic events:

\[
T_{TM}(\delta) = \left( \frac{\delta}{C_{TM}} \right)^{E_{TM}},
\]

The currency average scaling parameters \( E_{TM} \) and \( C_{TM} \) computed in \cite{Glattfelder et al. 2011} are 2.02 and \( 1.65 \times 10^{-3} \), correspondingly. Putting these coefficients into equation \ref{eq:17} one can calculate that thresholds reciprocal to the selected time intervals are: \( \delta_1 = 0.013\% \), \( \delta_2 = 0.039\% \), \( \delta_3 = 0.095\% \), and \( \delta_4 = 0.458\% \). It is worth mentioning that applied scaling parameters are relevant only to the FX market which was the object of the research in \cite{Glattfelder et al. 2011}. To the extent of our knowledge, parameters specific to Bitcoin prices were not mentioned in the scientific literature before. Therefore, as the first step, we obtained the parameters by studying the ‘time of total-move’ scaling law of historical Bitcoin returns. In figure \ref{fig:3(a)} the log-log plot of waiting times \( T_{TM}(\delta) \) is provided. The red line marks BTC/USD scaling law and is shown together with black and green lines computed for EUR/USD and Geometrical Brownian Motion (GBM). Settings of the latter are chosen to mimic realistic returns typical for the FX market.

Scaling law parameters, obtained in the experiment, exhibit distinct resemblance of the stylized properties of the traditional FX and the emerging Bitcoin markets. Scaling factors \( E_{TM} \) of EUR/USD and BTC/USD are 1.827 and 1.818 correspondingly (\( \approx 0.5\% \) difference). The same scaling factor of the GBM is 1.920 (\( \approx 5.6\% \) difference with EUR/USD) which is noticeably distant from the parameters of the analysed exchange rates. This divergence is present due to the non-normal distribution of real returns at ultra-short timescales (fat tails). The effect is pronounced in figure \ref{fig:3(a)} as the upward bend of the curves towards the beginning of the x-axis. Linear regressions, built in the range of straight parts of the curves, return scaling coefficients \( E_{TM} \) of real exchange rates which are very close to the ones observed in GBM. This evidence is an additional confirmation of the ‘Aggregational Gaussianity’ stylized fact typical for high-frequency markets \cite{Cont 2001}. Scaling parameters \( C_{TM} \) of EUR/USD and BTC/USD are 9.07e − 4 and 28.35e − 4 correspondingly. These values are significantly different because of the unlikely scale of the Bitcoin and EUR volatility and are not critical for the analysis.

Scaling law parameters \( E_{TM} \) and \( C_{TM} \) found for BTC/USD prices were used to compute the size of directional-change thresholds which would in an average register the number of intrinsic events equal to the number of periods \( n_k \). Expressing the parameter \( \delta_k \) from the equation \ref{eq:17} we
find that for BTC/USD the thresholds are: $\delta_1 = 0.09\%$, $\delta_2 = 0.33\%$, $\delta_3 = 0.89\%$, $\delta_4 = 5.13\%$. The values are about 10 times bigger than the ones for the FX market because of the proportionally larger realised volatility.

The set of selected time intervals and the corresponding thresholds for two markets were applied to compute realised and instantaneous volatility with the traditional and the novel approach. In Table 1 we present: average value of the realised volatility $\langle \sigma_{trad} \rangle$ computed as the sum of all four measurements with $k = \{1, 2, 3, 4\}$ divided by the number of experiments; its standard deviation $\sigma_{trad}$; average value of the instantaneous volatility computed by the novel approach $\langle \sigma_{DC} \rangle$; the corresponding standard deviation $\sigma_{DC}$; ratio of both measures $\langle \sigma_{trad} \rangle / \langle \sigma_{DC} \rangle$ and $\sigma_{trad} / \sigma_{DC}$. The last column of the table demonstrates the difference in the stability of results obtained by two measures. The gap between the sizes of the realised and the instantaneous volatility is significant and is pronounced across all tested exchange rates. The volatility computed in the ‘natural’ way persistently exceeds the instantaneous volatility discovered via the novel approach. The divergence grows up to 1.40 times in case of EUR/JPY. Two types of Bitcoin’s volatility appear to be only 5% different (column $\langle \sigma_{trad} \rangle / \langle \sigma_{DC} \rangle$). This phenomenon is captivating especially taking into account that Bitcoin is particularly famous due to its oversized price activity. Its activity is clearly pronounced in the large standard deviation of the BTC/USD pair. FX exchange rates, having noticeably smaller realised volatility, are characterized by the wider range of the standard deviation values. The ratio $\sigma_{trad} / \sigma_{DC}$ reaches the 0.02 level computed for EUR/USD.

4.3. Discrete price impact

The standard deviation computed for four different directional-change thresholds has extremely high value. This indicates that in contrast to the realised volatility the instantaneous one does not scale together with time. The cause of the detected aspect is the price discontinuity typical for all real markets. Conventional exchange architecture restricts the price quotations to be a multiple of some constant, for example, 0.001 of a USD. This fact caused substantial debates in the scientific
instead of 12 million ticks for 26 exchange rates, we have in average 100 million ticks for each FX
creased granularity thanks to the more detailed historical time series employed for the experiment:

\[ \sigma \text{ is built around the continuous Brownian motion and connects the number of directional changes} \]

\[ \text{with the instantaneous volatility. It has no adjustment factors to the discreteness of the analysed} \]

\[ \text{process. The directional-change intrinsic time does not precisely tick at the level where the size of} \]

\[ \text{the return is equal to the size of the threshold } \delta \text{ if the size of a price step is relatively big. Instead, in} \]

\[ \text{most cases, a new directional-change event becomes registered when the price has already jumped} \]

\[ \text{over the expected level (slippage). That is, two factors contribute to the size of the instantaneous} \]

\[ \text{volatility computed by the novel approach: the scale of the selected threshold } \delta \text{ and the tick size} \]

\[ \text{in the given data sample (discreteness).} \]

We performed the following experiment to estimate the impact of the price discreteness and the
threshold size \( \delta \) on the computed instantaneous volatility. Three time series were generated by
GBM with the various density of ticks per period of time. Variation of the number of price changes
in the simulation is equivalent to changing the simulated tick size having fixed a one-year time
interval and volatility of the generated process equal to 15%. Forty directional-change thresholds
with fixed increment starting from 0.01% and ending at 0.29% were applied to all three GBMs. We
provide computed instantaneous volatility of simulated time series in figure 3(b). Two particular
properties can be noticed. First, one brings the generated time series closer to the continuous process
by making the size of a tick smaller (increasing the number of price changes in the sample). In
this case, the values of the estimated instantaneous volatility \( \sigma_{DC} \) become closer to the volatility \( \sigma \)
embedded in the model. Second, bigger thresholds are less sensitive to the discreteness of the given
set of prices. The slippage effect becomes less pronounced and the obtained result also approaches
the value \( \sigma \) when the tick size represents a small fraction of the directional-change thresholds. Both
effects emphasise the dissimilarity of realised and instantaneous volatilities. A more comprehensive
analysis should be performed to bridge the gap between these two measures.

### 4.4. Volatility seasonality

The seasonality of realised volatility in the traditional markets is a relatively well-studied topic.
\[ \text{Dacorogna et al. (1993) presented a weekly seasonality pattern of price activity in the FX market.} \]

Their analysis depends on the assumption that trading happens at various time zones and within
specific trading hours. Such a physical distribution of traders is embodied in geographical compo-
nents of the market activity. We do not build on a similar assumption in our work. The collection
of observed historical returns is treated as the only source of available for the analysis information.

We divide a whole week into a set of 10-minute time intervals (bins) for each of which the average
number of directional changes will be computed to find the seasonality pattern of instantaneous
volatility typical for Bitcoin prices and FX exchange rates. There are 1008 equally spaced points
located on the fixed distance from the beginning of each week in total. This is significantly large
number than the one used in the work \[ \text{Dacorogna et al. (1993) (168 points). We can afford this} \]

decreased granularity thanks to the more detailed historical time series employed for the experiment:
instead of 12 million ticks for 26 exchange rates, we have in average 100 million ticks for each FX

<table>
<thead>
<tr>
<th>Name</th>
<th>( \langle \sigma_{trad} \rangle ), %</th>
<th>( \sigma_{trad} )</th>
<th>( \langle \sigma_{DC} \rangle ), %</th>
<th>( \sigma_{DC} )</th>
<th>( \langle \sigma_{trad} \rangle / \langle \sigma_{DC} \rangle )</th>
<th>( \sigma_{trad} / \sigma_{DC} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>EUR/USD</td>
<td>9.72</td>
<td>0.03</td>
<td>7.53</td>
<td>1.38</td>
<td>1.29</td>
<td>0.02</td>
</tr>
<tr>
<td>EUR/JPY</td>
<td>11.93</td>
<td>0.12</td>
<td>8.55</td>
<td>2.07</td>
<td>1.40</td>
<td>0.06</td>
</tr>
<tr>
<td>EUR/GBP</td>
<td>8.04</td>
<td>0.23</td>
<td>5.81</td>
<td>1.43</td>
<td>1.38</td>
<td>0.16</td>
</tr>
<tr>
<td>BTC/USD</td>
<td>84.76</td>
<td>8.67</td>
<td>80.87</td>
<td>22.21</td>
<td>1.05</td>
<td>0.39</td>
</tr>
</tbody>
</table>

Table 1. Volatility computed using the ‘traditional’ (equation 16) and the directional-change approaches (equation 15). Provided values \( \langle \sigma_{trad} \rangle \) and \( \langle \sigma_{DC} \rangle \) are the average of four measurements performed with specific parameters: in the ‘traditional’ case time intervals between observations \( S_n \) and \( S_{n-1} \) are 1 minute, 10 minutes, 1 hour, and 1 day and in case of the novel approach the thresholds \( \delta \) are \( \delta_1 = 0.013\% \), \( \delta_2 = 0.039\% \), \( \delta_3 = 0.095\% \), \( \delta_4 = 0.458\% \) (FX prices) and \( \delta_1 = 0.09\% \), \( \delta_2 = 0.33\% \), \( \delta_3 = 0.89\% \), \( \delta_4 = 5.13\% \) (BTC prices).
We selected the threshold $\delta = 0.01\%$ for the first experiment with FX exchange rates. The average number of directional changes in a week registered by a threshold of this size is approximately equal to the number of bins in it. The computation of the seasonality pattern is done in the following steps. First, we run all historical tick-by-tick prices through the directional-change algorithm with the specified threshold $\delta$. As soon as a new intrinsic event becomes registered we check within which bin it happened and add +1 to the number of directional changes corresponding to this time interval. When there are no prices left in the historical time series, we find the average number of intrinsic events per each bin and apply equation 15 to compute the corresponding instantaneous volatility. Considering the five-year-long historical data, the obtained average is based on nearly 250 observations. Calculated values should be later normalised by the number of years to get the annualised volatility.

The reconstructed instantaneous volatility seasonality pattern of the FX pairs is shown in figure 4. The pattern is notably stable across all tested exchange rates and is similar to the one demonstrated in Dacorogna et al. (1993). This similarity confirms the idea that the seasonality of instantaneous volatility is subject to the geographical distributing of the trading centres across the world.

We provide results of the same experiment where the ‘traditional’ volatility estimator (equation 16) was employed to reveal the seasonality patterns of the FX exchange rates in Appendix E. The ‘traditional’ pattern is less affected by the source of the given time series. The average difference between the realised volatility of the most active pair (EUR/JPY) and the least active (EUR/GBP) is equal to 46%. The difference in the volatility computed by the novel approach rises to 56%. The vertical distance between EUR/USD and EUR/GBP curves also becomes less pronounced computed by the ‘natural’ estimator. At the same time, this estimator demonstrates more rapid changes between values of each couple of consecutive bins. Local maximums at the beginning and the end of a day are considerably abrupt. The reason of this is that the directional-change intrinsic time captures the part of the volatility free of the noise component of the underlying process. The exact form and scale of this part is a topic for the future research work.

Assets traded in the crypto market have several specific properties which make them noticeably distinct from the traditional financial instruments such as FX exchange rates. Among the charac-
characteristics are open trading within weekends and holidays; the absence of isolated physical trading centres where working hours are fixed; still low acceptance of the emerging market among professional traders. We explored the seasonality pattern of Bitcoin instantaneous volatility to check whether these specialities have any considerable impact on its shape behaviour. The seasonality pattern is presented in figure 5. We apply the same threshold size $\delta = 0.01\%$ used in the previous experiment to compare the seasonality patterns of Bitcoin and EUR. In contrast to EUR/USD, the periodical shape of Bitcoin’s curve is much less pronounced. Its standard deviation computed within a week is 0.5\% which is roughly four times smaller than the standard deviation of the EUR/USD pattern equal to 1.9\%. Surprisingly, the intra-day maximums and minimums do not precisely coincide with those observed in the traditional markets. They are shifted towards the time intervals where European and American markets contribute the most to the geographical pattern disclosed in Dacorogna et al. (1993). This observation confirms the one provided in Eross et al. (2017). Realised volatility over weekends is slightly lower than the middle part of the week and is practically equal to the Monday’s activity. We attribute the observed facts to the mentioned above non-traditional characteristic of the cryptocurrency market.

Computed by the novel approach instantaneous volatility directly depends on the size of the selected $\delta$ (figure 3(b)). To examine the impact of the threshold size we arbitrary selected the following set of values $\delta = \{0.01\%, 0.04\%, 0.10\%\}$. The same algorithm described above was applied to reconstruct the volatility seasonality pattern for the FX pair with the highest daily trading volume: EUR/USD (BIS 2016). The seasonality patterns shift toward higher volatility values when the size of the threshold is bigger which is in line with the results of the experiments on GBM (figure 3(b)). Average values of the seasonality curves computed with thresholds $\delta$ equal to 0.10\% and 0.04\% are 1.71 and 1.57 times higher than the values computed with $\delta = 0.01\%$. The difference in the amplitude of all patterns is even more pronounced: the seasonality curve constructed with the smallest in the set threshold is much sleeker (less wander) than the rest of the curves. This phenomenon should urge researchers and practitioners to select threshold according to their needs very carefully employing the directional-change technique.

According to Table 1 realised volatility of Bitcoin returns computed in the ‘traditional’ way is about nine times bigger than the analogous volatility of the FX exchange rates. Besides, the retrieved sample of historical BTC/USD prices has 1.2 million ticks per year which is 16.7 times smaller than the number of ticks per year in the EUR/USD case (about 20 millions). As a result, the choice of the directional-change threshold $\delta$ has a much more significant effect on the average
value of the BTC/USD instantaneous volatility. We demonstrate results of four experiments with different threshold sizes in figure 7. The same \( \delta = 0.01\% \) is used as the reference for the set of all thresholds: \( \delta = \{0.01\%, 0.03\%, 0.10\%, 0.20\%\} \). As it can be seen from figure 7, the increase in the size of \( \delta \) causes the corresponding increase in the volatility level around which the seasonality pattern oscillates. The levels of the seasonality distribution for \( \delta = \{0.03\%, 0.10\%, 0.20\%\} \) are 1.5, 4.0, and 11.1 times bigger than the value corresponding to the smallest threshold \( \delta = 0.01\% \). The biggest \( \delta = 0.20\% \) lifts the value up to the level of 68.5\% (which is still smaller than the realised volatility presented in Table 1 (84.76\%)).

4.5. Autocorrelation and theta-time

Persistent seasonality patterns of the instantaneous volatility computed for the FX exchange rates have a shape which changes with clear daily periodicity. This observation suggests that there should be a strong autocorrelation of the instantaneous volatility or, in other words, of the number
of directional changes. To check this statement, we examined the autocorrelation function (ACF) of the number of directional changes observed within each bin of a week. Results of the experiment made for the FX exchange rates are provided in figure 8. It is not wrong to say that figure 8 also indicates the ACF of the volatility since the number of directional changes is directly related to the instantaneous volatility through the equation \(15\). The same size of the directional-change threshold used to reveal the seasonality distribution \(\delta = 0.01\%\) was employed. A remarkably stable pattern was found where daily and weekly seasonality is easily recognisable. Although the ACF function of FX exchange rates discovered in our work is highly similar to results provided by Dacorogna et al. (1993), there are clear differences between the two patterns. The ACF of the number of directional changes computed through time lags defined in physical time does not cross the zero level for a much more extended period and is consistently positive with lags even greater than several weeks. The curve representing the ACF of EUR/JPY has the smallest amplitude (smallest variability). In contrast, curves of EUR/USD together with EUR/GBP invariably follow the same pattern shifted up in the case of EUR/USD.

The BTC/USD exchange rate characterised by much less pronounced seasonality pattern of the instantaneous volatility has also been tested by the autocorrelation function. The results are presented in figure 8. The main difference between the values computed for two markets is the amplitude of the ACF curves: the variability of the BTC/USD curve is 10 times lower than the variability of the EUR/USD curve. We note that significantly bigger thresholds have also been tested, but they reveal less accurate patterns due to insufficiently frequent data for the statistical analysis.

It can be seen from figure 8 that the ACF of the FX exchange rates exhibits a certain level of decline. The large seasonal peaks of the autocorrelation functions in physical time do not allow to measure its decay precisely. A measure capable of converting the price evolution process to the stationary one should be applied to better estimate the level of the downturn. We removed the seasonality pattern by employing the concept of theta time (\(\Theta\)-time) proposed by Dacorogna et al. (1993). \(\Theta\)-time is designed to eliminate the periodicity pattern by defining a set of non-equal intervals within which the measure should be performed. The length of each interval in \(\Theta\)-time depends on the historical activity of the market rather than on equally spaced periods of the homogeneous physical time. Therefore, the cumulative price activity (volatility) between each consecutive couple of \(\Theta\) steps is constant. The distance between \(\Theta\) timestamps, measured in physical time, is dictated by the shape of the volatility seasonality pattern and, in contrast to
physical time initially used to reveal the pattern, is not a fixed value. Periods of high activity are equivalent to shrinking the speed of physical time and the frequency of \( \Theta \) stamps increases. In contrast, periods of low activity are identical to stretching the flow of the physical time and the lower number of \( \Theta \) intervals appears. Active parts of the seasonality pattern have the higher density of \( \Theta \) timestamps per a unite of the physical time than the standstill sections. Mathematically:

\[
\Theta(t) = \int_{t_0}^{t} \sigma(t')dt',
\]

where \( t_0 \) and \( t \) are the beginning and the end of the considered period of physical time and \( \sigma(t') \) is the value of the instantaneous volatility corresponding to each moment of the interval. Equation 18 transforms into the sum of elements \( \sigma_{\Delta t'} \) between the beginning and the end of the observed interval \( \Delta t_0 \) and \( \Delta t \) in case of non-continuous seasonality pattern where the values are discretely defined in periods \( \Delta t \) (as in our experiment):

\[
\Theta(t) = \sum_{\Delta t_0}^{\Delta t} \sigma_{\Delta t'}.
\]

It should be noted that the number of bins in a week is always constant in both physical and \( \Theta \) times. This is achieved through the assumption that the integral (or the sum) of the weekly activity is the constant value.

The autocorrelation function of the number of directional changes computed in \( \Theta \)-time is shown in Figures 9(a) and 9(b) in normal and log-log scales. Curves on the log-log plot outline the first 100 bins and are approximated by straight lines corresponding to the scaling law with coefficients \( E_{ACF} \) and \( C_{ACF} \) provided in Table 2. Major weekly fluctuations of the volatility seasonality pattern have been successfully eliminated. Nevertheless, the picture has several peculiar properties which should be discussed. First, \( \Theta \)-time does not completely remove the seasonality shape of ACF in the same way it happened in the work Dacorogna et al. (1993): noticeable peaks are still present in the final part of each business day. The similar phenomenon, observed in the original paper, was explained by the non-optimal setup of the chosen model; it assumed the same activity for all working days, which is, indeed, not fully correct (see figure 4). However, we do not use any analytical expression postulating the equal daily activity to describe the seasonality pattern. Instead, components \( \sigma_{\Delta t'} \) of real empirically found volatility seasonality patterns depicted in Figures 4 and 5 were utilised to define the timestamps in \( \Theta \)-time. Therefore, we eliminate the inefficiency connected to the assumption mentioned above which means an alternative interpretation for the remained seasonality should be provided.

We attribute the remained fluctuations to the algorithm of the price dissection into a collection of alternating trends. We also claim that the choice of the number of bins in the experiment affects the shape of the autocorrelation function in theta time. The dissection procedure has to be initialised only once and then performs unsupervised. The evolution of the price curve dictates the sequence of intrinsic events. This fact leads to the certain dilemma: to which bin of a week the intrinsic event should be assigned? A couple of prices, at which two subsequent directional changes become registered, could belong to a different bins. Let us say these are the intervals \( \Delta t_{n-1} \) and \( \Delta t_n \). This means that the beginning of the price move that eventually reached the level of the latest intrinsic event had started within \( \Delta t_{n-1} \). The end of this price trajectory finishes within the interval \( \Delta t_n \). The crucial point is at what part of the \( \Delta t \) this beginning and end are located. In the extreme case, the whole price trajectory before the directional change could be fully placed inside of the interval \( \Delta t_{n-1} \). The latest tick that eventually triggered the new directional-change event can be at the very beginning of \( \Delta t_n \). Should such an event be assigned to the bin \( \Delta t_{n-1} \) or to \( \Delta t_n \)? The answer to this question is particularly important considering the effect shown in figure 6. The inconsistency affects the seasonality pattern by not only changing its average amplitude but also by providing
Table 2. Parameters of the scaling law describing the exponential decay of the ACF of the number of directional changes in Θ-time (figure 9(b)).

<table>
<thead>
<tr>
<th>Name</th>
<th>$C_{ACF}$</th>
<th>$E_{ACF}$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BTC/USD</td>
<td>0.038</td>
<td>-0.045</td>
<td>-0.990</td>
</tr>
<tr>
<td>EUR/USD</td>
<td>0.026</td>
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<td>-0.975</td>
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<td>EUR/JPY</td>
<td>0.011</td>
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</tr>
<tr>
<td>EUR/GBP</td>
<td>0.001</td>
<td>-0.059</td>
<td>-0.953</td>
</tr>
</tbody>
</table>

slightly different regions of low and high instantaneous volatility (see, for example, the curves for $\delta = 0.01\%$ and $\delta = 0.10\%$).

A better way of associating locations of intrinsic events with bins of a week is another question related to the transition from the physical to intrinsic time and vice versa. This topic should be discussed in more details in further research works. Until then, a potential strategy to resolve the localisation problem is the use of smaller thresholds and bigger time intervals.

5. Concluding remarks

Analytical solutions presented in this paper translate the language of traditional risk-management tools based on drawdowns and drawups into the language of the directional-change intrinsic time. This connection makes it possible to interpret the evolution of a price curve as a sequence of alternating trends of the given scale. The observed number of directional changes for a period of time has been connected to the properties of the studied time series characterised by the instantaneous volatility $\sigma_{DC}$ and the trend $\mu$. The choice of directional-change thresholds $\delta_{up}$ and $\delta_{down}$ used to dissect the historical price curve is arbitrary but affects the results of the experiment. Equations 12, 6, and 7 have been confirmed by a Monte Carlo simulation which demonstrates the robustness and accuracy of the obtained analytical expressions.

We extended the work of Dacorogna et al. (1993) through discovering the weekly seasonality pattern of the instantaneous volatility of three FX exchange rates as well as of Bitcoin prices. The connection of the number of directional-change intrinsic events to the instantaneous volatility has been employed to perform the computation. Similar patterns of the realised and the instantaneous
volatilities were obtained. Several noticeable differences between the results demonstrated in the work [Dacorogna et al. (1993)] and the ones presented in the current paper have been highlighted. First, the novel method significantly simplifies the construction of the instantaneous volatility seasonality pattern using tick-by-tick prices. Second, the autocorrelation function of the number of directional changes computed in physical time stays positive for a notably long period of time. Third, the beginning of the autocorrelation function computed in Θ-time can be approximated by the exponential function when the rest of it declines linearly.

The number of directional changes, directly connected to the instantaneous volatility of the given discrete time series, reveals the differences between scales of measures utilised to study the evolution of the price curve. The proposed framework represents a novel conceptual paradigm where measures are independent of the flow of physical time. We argue that this characteristic has significant advantages over ‘natural’ volatility estimators and has to be considered as the primary tool in the set of classical risk management techniques.

The insights provided within this paper underline the relevance of the proposed directional-change framework as a valuable alternative to the traditional time-series analysis tools. The directional-change intrinsic time has the remarkable ability to efficiently deal with fat-tailed distribution of returns and is more efficient in capturing periods of changing price activity. Results of the provided research extend the set of risk management tools constructed to evaluate the statistical properties of traditional and emerging financial markets.

**References**


Harris, L., Trading and exchanges: Market microstructure for practitioners, 2003, Oxford University Press, USA.


Appendix A: Drawdowns and drawups

Probabilities of drawdowns and drawups were extensively studied by Carr et al. (2011) to propose a new insurance technique against unexpected price moves and a novel way of hedging liabilities associated with these risks. Zhang and Hadjiliadis (2012) employed drawdowns as an estimate of the stock default risk and also provided a risk-management mechanism affecting the investor’s optimal cancellation timing. In Schuhmacher and Eling (2011) drawdowns are considered as one of 14 reward-to-risk ratios alternative to the widely known performance measure such as the Sharpe ratio. Properties of drawdowns can be applied as an estimate of the portfolio optimisation problem (Grossman and Zhou 1993, Chekhlov et al. 2005). The latter can be personalised to match traders’ or investors’ expectations and the tolerance to the size and the length of the market disruption.

Drawdowns $D_t$ and drawups $U_t$, also called rallies (Hadjiliadis and Večer 2006), registered by the moment of time $t$ depend on the running price maxima $P_t$ and the running price minima $P_0$ (Zhang and Hadjiliadis 2012, Mijatović and Pistorius 2012, Landriault et al. 2015). These reference points hinge on the set of historical prices $P_s$ and are mathematically defined in the following way:

$$P_t = \sup\{P_s : 0 \leq s \leq t\} \quad \text{and} \quad P_0 = \inf\{P_s : 0 \leq s \leq t\}, \quad (A1)$$

where $t \geq 0$ and the interval $[0, t]$ is fixed. Drawdowns and drawups are the differences between the final price of the given time interval $P_t$ and the registered local maxima and minima:

$$D_t = P_t - P_t \quad \text{and} \quad U_t = P_t - P_0. \quad (A2)$$

Once a price curve experiences a drawdown $D_t$ of the size $a$, the waiting time $\tau^D_a$ is registered. Similarly, for a drawup the waiting time associated with the size $a$ is $\tau^U_a$. In details:

$$\tau^D_a = \inf\{D_t \geq a : t \geq 0\} \quad \text{and} \quad \tau^U_a = \inf\{U_t \geq a : t \geq 0\}. \quad (A3)$$

Appendix B: Intrinsic time

All relevant to the performance of the financial system events such as political decisions, natural disasters, or economic reports rarely happen synchronously or are equally spaced in time. A sequence of them has a non-homogeneous nature and is not characterised by any vital autocorrelation function. Ultimately, the change of days and nights, as well as seasons, is dictated by the natural structure of the physical world which is barely connected with the flow of financial activity. Human minds with the whole diversity of peculiar and indescribable characteristics are primal engines of all market’s evolutionary shifts. The global market, where the majority of transactions happen online and where traders, dealers, and market makers are distributed all around the world, is completely blind and deaf to the periodicity of days and nights as well as to the climate factors of any standalone region of the Earth.

Guillaume et al. (1997) provided the concept of alternating directional changes capable to connect the continuous flow of physical time with the evolution of price returns. In this event-based space, only a sequence of prices at which directional changes of the given size become observed and corresponding local extremes describe changes of the system’s states. Thus, the set of intrinsic events is decoupled from the flow of physical time. An example of a price curve dissected by the directional-change algorithm is provided in figure B1. The density of intrinsic events depends only on the evolution of the price curve. The latter is pronounced in different number of events registered within intervals of equal length. Thus, only the end of the section 1 is located in the period $T_1$ while $T_2$ contains ends of three sections, 2, 3, 4, and $T_3$ holds ends of two sections, 5 and 6. This property of directional-change intrinsic time allows to efficiently capture the most relevant to risk management information: tipping points of trend changes. At the same time, it
possesses properties of a filtering technique by ignoring price changes between directional changes. In contrast, equally spaced through periods $T_1, T_2, T_3$ prices extracted from the same curve do not contain information on its extreme activity located in $T_2$. This disability of the traditional techniques over stochasticity of market’s speed develops into too stiff volatility estimators.

Specific properties of historical prices can be described studying a collection of directional changes. Guillaume et al. (1997) was the first researcher to uncover a scaling law$^1$ relating the expected number of directional-changes $N(\delta)$ to the size of the threshold $\delta$. Mathematically:

$$N(\delta) = \left( \frac{\delta}{C_{N,DC}} \right)^{E_{N,DC}},$$  

(B1)

where $C_{N,DC}$ and $E_{N,DC}$ are the scaling law coefficients. Later, Glattfelder et al. (2011) employed the framework to discover 12 independent scaling laws which hold across three orders of magnitude and are present in 13 currency exchange rates. The persistence of revealed scaling laws became the base elements for the tools designed to monitor market’s liquidity at multiple scales (Golub et al. 2014). Later Golub et al. (2014) described a successful trading strategy exploiting a collection of tools build upon the directional-change intrinsic time and characterised by the annual Sharp ratio greater than 3.0.

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$^1$ A basic polynomial functional relationship where a change in input results in a proportional change in output.
Appendix C: Bitcoin seasonality

Despite the relatively young age of the blockchain technology, there are already a few research works concern the statistical properties of cryptocurrencies. Sapuric and Kokkinaki (2014) analysed realised volatility of Bitcoin returns within 4-years time interval to understand what are the prime characteristics of its price activity. They confirmed Bitcoin’s high volatility, but emphasised that traded volume is a remarkable fact which should be taking into account computing the realised volatility. The authors compare Bitcoin with conventional financial instruments, including gold, and several national currencies and demonstrate that the calculated volatility significantly decreases when the traded volumes are included in the model.

Haferkorn and Diaz (2014) studied seasonality patterns of the number of payments performed in three cryptocurrencies: Bitcoin (classified as a worldwide payment system), Litecoin (open source software project), and Namecoin (decentralised name system). Their research confirmed that in contrast to the traditional equity and FX markets, the monthly or yearly seasonality is not typical for the crypto market. The only robust weekly pattern was fund in Bitcoin prices while Litecoin and Namecoin had weak or no patterns at all. The authors state that there is also no significant correlation between the returns of observed exchange rates. Authors speculate that the reason for this is the fact that although these cryptocurrencies have similar core architecture, they all have been created to serve quite specific needs.

JE de Vries and Aalborg (2017) made another attempt to discover seasonality patterns of Bitcoin prices analysing daily traded volume, daily transaction volume, and Google trends (the number of searches for the word ‘bitcoin’). The author also inspected the seasonality of the number of transactions performed from and to individual blockchain accounts. All of the measurements demonstrated no particular periodicity.

Eross et al. (2017) gave a more affirmative answer on the existence of the intraday seasonality of Bitcoin prices. The authors investigated Bitcoin returns, volume, realised volatility, and bid-ask spreads to reveal several intraday stylized facts. A significant negative correlation was found between returns and volatility, while volume and volatility have a considerably positive correlation. The authors attribute such patterns to the European and North American traders as well as to the lack of market makers in the whole crypto space.

Appendix D: Monte Carlo waiting times

Appendix E: ‘Traditional’ volatility seasonality

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2 According to the Bank for International Settlements the daily average FX trading volume in April 2016 was $5.1 trillion (BIS 2016) when the highest registered volume in the crypto market is only $45.8 billion (https://coinmarketcap.com/charts/).
Table D1. Waiting times and number of directional changes in a Monte Carlo simulation. $\mu$ and $\sigma$ are parameters of the Brownian motion used for the test. $10^9$ ticks in the simulated time series. $N_{MC}$, $\langle T_{up}^{MC} \rangle$, and $\langle T_{down}^{MC} \rangle$ are the number of directional changes and the average waiting times registered in the Monte Carlo simulation. $E[N_{DC}]$, $E[T_{up}]$, and $E[T_{down}]$ are theoretical values dictated by equations 12, 6, and 7 correspondingly. Values $\sigma_{T_{up}}^{MC}$ and $\sigma_{T_{down}}^{MC}$ are standard deviations of empirical and theoretical waiting times.

<table>
<thead>
<tr>
<th>$\mu$, %</th>
<th>$\sigma$, %</th>
<th>$N_{DC}^{MC}/E[N_{DC}]$</th>
<th>$\langle T_{up}^{MC} \rangle/E[T_{up}]$</th>
<th>$\sigma_{T_{up}}^{MC}$</th>
<th>$\langle T_{down}^{MC} \rangle/E[T_{down}]$</th>
<th>$\sigma_{T_{down}}^{MC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>20</td>
<td>1.028</td>
<td>0.968</td>
<td>2.54e-05</td>
<td>1.019</td>
<td>2.53e-06</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>1.009</td>
<td>0.989</td>
<td>2.78e-06</td>
<td>1.012</td>
<td>3.32e-07</td>
</tr>
<tr>
<td>30</td>
<td>60</td>
<td>1.001</td>
<td>0.995</td>
<td>8.79e-07</td>
<td>1.033</td>
<td>9.58e-08</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>1.021</td>
<td>0.971</td>
<td>2.29e-05</td>
<td>1.043</td>
<td>2.59e-06</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
<td>1.005</td>
<td>0.993</td>
<td>2.94e-06</td>
<td>1.019</td>
<td>3.29e-07</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>0.987</td>
<td>1.011</td>
<td>8.84e-07</td>
<td>1.034</td>
<td>9.98e-08</td>
</tr>
<tr>
<td>11</td>
<td>20</td>
<td>1.029</td>
<td>0.968</td>
<td>2.20e-05</td>
<td>1.011</td>
<td>2.78e-06</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
<td>0.994</td>
<td>1.006</td>
<td>2.72e-06</td>
<td>0.997</td>
<td>3.30e-07</td>
</tr>
</tbody>
</table>

Figure E1. realised volatility seasonality patterns of three FX exchange rates computed using the traditional approach (equation 16). 1 minute intervals have been used to calculate returns. The size of each bin is 10 minutes, 1008 bins in total.
Figure E2. Realised volatility seasonality patterns of BTC/USD and EUR/USD exchange rates computed using the traditional approach (equation 16). 1 minute intervals have been used to calculate returns. The size of each bin is 10 minutes, 1008 bins in total.
Scaling Laws in Bitcoin and Forex Markets

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†Department of Banking and Finance, University of Zurich

Abstract

In this work, we investigate the presence and properties of scaling laws in the Bitcoin market. The set of laws corresponds to 12 independent and eight additional properties of the Forex market initially described by Glattfelder et al. (2011). For the first time, we demonstrate that the laws are among characteristics of the Bitcoin/American-Dollar (BTC/USD) tick-by-tick prices despite the different nature and mechanics of the Forex and cryptocurrency markets. However, revealed dependencies have significant deviations from the straight line in the range of small values. The latter is addressed to the significantly higher volatility of the Bitcoin prices. Besides, we demonstrate the difference of the scaling laws discovered in the EUR/USD using the data from Glattfelder et al. (2011) (analysed interval from 2002 to 2007) and a new time series which corresponds to the period from 2015 to 2018.

1 Data

We selected the Bitcoincharts.com1 platform to download the historical information of all performed trades at one of the oldest crypto exchanges Coinbase2 within the time interval from 2015-06-15 to 2018-04-01. The total number of ticks in the downloaded sample is 21’330’645. The new set of the Forex data was obtained at the JForex trading platform from Dukascopy3 which provides a comprehensive list of all historical trades in the EUR/USD market. The covered time-span is from 2015-06-15 to 2018-06-21 and the corresponding number of ticks is 91’200’125. Both historical price curves are presented in Figure 1. The volatility of both exchange rates computed over the moving window of one week are depicted in Figure 2.

∗Corresponding author: vladimir.petrov@uzh.ch
1http://api.bitcoincharts.com/v1/csv/
2https://www.coinbase.com/
3https://www.dukascopy.com/swiss/english/forex/jforex/
Figure 1: Daily prices of Bitcoin and Euro versus American dollar (BTC/USD and EUR/USD correspondingly) over the three-year time interval.

Figure 2: Annualised volatility of BTC/USD and EUR/USD calculated over the one-week moving week.
Figure 3: Scaling laws of BTC/USD (orange line) and EUR/USD (grey line) exchange rates computed over the three-year interval. The order of insets coincides with the order proposed in Glattfelder et al. (2011).
Table 1: Estimated scaling law parameters $C$ and $E$ considering BTC/USD prices and their deviations $\Delta C,\%$ and $\Delta E,\%$ from the parameters corresponding EUR/USD exchange rates computed over the same period of time (2015-2018).

<table>
<thead>
<tr>
<th>Name</th>
<th>$C$</th>
<th>$\Delta C,%$</th>
<th>$E$</th>
<th>$\Delta E,%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean price move</td>
<td>1.37E+04</td>
<td>-98.39</td>
<td>0.55</td>
<td>3.24</td>
</tr>
<tr>
<td>Quadratic mean price move</td>
<td>4.42E+03</td>
<td>-99.00</td>
<td>0.48</td>
<td>-2.11</td>
</tr>
<tr>
<td>Directional change count</td>
<td>1.39E+02</td>
<td>1322.15</td>
<td>-1.81</td>
<td>-3.60</td>
</tr>
<tr>
<td>Price move count</td>
<td>1.20E+02</td>
<td>1273.46</td>
<td>-1.88</td>
<td>-4.51</td>
</tr>
<tr>
<td>Maximum price move</td>
<td>4.36E+03</td>
<td>-97.87</td>
<td>0.61</td>
<td>8.69</td>
</tr>
<tr>
<td>Quadratic mean max price move</td>
<td>1.52E+03</td>
<td>-98.92</td>
<td>0.53</td>
<td>2.41</td>
</tr>
<tr>
<td>Mean time of price move</td>
<td>1.22E-02</td>
<td>810.28</td>
<td>1.88</td>
<td>-4.43</td>
</tr>
<tr>
<td>Time during directional changes</td>
<td>1.01E-02</td>
<td>911.89</td>
<td>1.81</td>
<td>-3.56</td>
</tr>
<tr>
<td>Total price move</td>
<td>4.30E-01</td>
<td>-13.52</td>
<td>0.93</td>
<td>-3.42</td>
</tr>
<tr>
<td>Time of total move</td>
<td>1.01E-02</td>
<td>922.55</td>
<td>1.81</td>
<td>-3.33</td>
</tr>
<tr>
<td>Total-move tick count</td>
<td>1.87E-02</td>
<td>2190.56</td>
<td>1.83</td>
<td>-2.40</td>
</tr>
<tr>
<td>Coastline (cum. total move)</td>
<td>1.95E+05</td>
<td>27863.07</td>
<td>-0.89</td>
<td>-3.80</td>
</tr>
<tr>
<td>Directional-change move</td>
<td>7.21E-01</td>
<td>-27.34</td>
<td>0.88</td>
<td>-10.66</td>
</tr>
<tr>
<td>Time of directional change</td>
<td>1.53E-02</td>
<td>952.92</td>
<td>1.75</td>
<td>-4.06</td>
</tr>
<tr>
<td>Directional-change tick count</td>
<td>3.42E-02</td>
<td>2304.63</td>
<td>1.82</td>
<td>-3.12</td>
</tr>
<tr>
<td>Cumulative directional change</td>
<td>6.27E+04</td>
<td>15503.94</td>
<td>-0.94</td>
<td>4.11</td>
</tr>
<tr>
<td>Overshoot move</td>
<td>1.19E+00</td>
<td>14.92</td>
<td>0.99</td>
<td>5.15</td>
</tr>
<tr>
<td>Time of overshoot</td>
<td>1.41E-02</td>
<td>899.22</td>
<td>1.84</td>
<td>-3.49</td>
</tr>
<tr>
<td>Overshoot tick count</td>
<td>2.33E-02</td>
<td>2154.49</td>
<td>1.83</td>
<td>-2.15</td>
</tr>
<tr>
<td>Cumulative overshoot</td>
<td>1.61E+05</td>
<td>54365.96</td>
<td>-0.82</td>
<td>-12.37</td>
</tr>
</tbody>
</table>
Figure 4: Scaling laws computed considering EUR/USD tick-by-tick prices (grey line) collected over the time interval from 2015-06-15 to 2018-06-21 and scaling laws reconstructed using the scaling coefficients from Glattfelder et al. (2011) (green line).
Table 2: Estimated scaling law parameters $C$ and $E$ considering the updated set of EUR/USD prices and their deviations $\Delta C,\%$ and $\Delta E,\%$ from the parameters provided in Glattfelder et al. (2011).

<table>
<thead>
<tr>
<th>Name</th>
<th>$C$</th>
<th>$\Delta C,%$</th>
<th>$E$</th>
<th>$\Delta E,%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean price move</td>
<td>8.54E+05</td>
<td>28.72</td>
<td>0.53</td>
<td>7.33</td>
</tr>
<tr>
<td>Quadratic mean price move</td>
<td>4.44E+05</td>
<td>15.59</td>
<td>0.49</td>
<td>3.78</td>
</tr>
<tr>
<td>Directional change count</td>
<td>9.74E+00</td>
<td>3.43</td>
<td>-1.88</td>
<td>-1.34</td>
</tr>
<tr>
<td>Price move count</td>
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<td>-7.71</td>
<td>-1.97</td>
<td>1.95</td>
</tr>
<tr>
<td>Maximum price move</td>
<td>2.05E+05</td>
<td>22.28</td>
<td>0.56</td>
<td>6.92</td>
</tr>
<tr>
<td>Quadratic mean max price move</td>
<td>1.41E+05</td>
<td>5.35</td>
<td>0.51</td>
<td>3.88</td>
</tr>
<tr>
<td>Mean time of price move</td>
<td>1.34E-03</td>
<td>9.37</td>
<td>1.96</td>
<td>1.89</td>
</tr>
<tr>
<td>Time during directional changes</td>
<td>1.00E-03</td>
<td>-38.81</td>
<td>1.88</td>
<td>1.71</td>
</tr>
<tr>
<td>Total price move</td>
<td>4.97E-01</td>
<td>2.07</td>
<td>0.96</td>
<td>-1.85</td>
</tr>
<tr>
<td>Time of total move</td>
<td>9.89E-04</td>
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<td>1.87</td>
<td>-0.73</td>
</tr>
<tr>
<td>Total-move tick count</td>
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<td>-1.07</td>
</tr>
<tr>
<td>Coastline (cum. total move)</td>
<td>6.97E+02</td>
<td>247.19</td>
<td>-0.92</td>
<td>-1.33</td>
</tr>
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<td>Directional-change move</td>
<td>9.93E-01</td>
<td>0.09</td>
<td>0.98</td>
<td>2.54</td>
</tr>
<tr>
<td>Time of directional change</td>
<td>1.45E-03</td>
<td>-11.19</td>
<td>1.83</td>
<td>-1.14</td>
</tr>
<tr>
<td>Directional-change tick count</td>
<td>1.42E-03</td>
<td>-96.60</td>
<td>1.88</td>
<td>-6.86</td>
</tr>
<tr>
<td>Cumulative directional change</td>
<td>4.02E+02</td>
<td>356.34</td>
<td>-0.90</td>
<td>-5.76</td>
</tr>
<tr>
<td>Overshoot move</td>
<td>1.03E+00</td>
<td>4.70</td>
<td>0.94</td>
<td>-5.41</td>
</tr>
<tr>
<td>Time of overshoot</td>
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<td>0.14</td>
</tr>
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<td>Overshoot tick count</td>
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</tr>
<tr>
<td>Cumulative overshoot</td>
<td>2.96E+02</td>
<td>179.32</td>
<td>-0.94</td>
<td>2.54</td>
</tr>
</tbody>
</table>
References

Bridging the Gap Between Physical and Intrinsic Time

Anton Golub∗1, Vladimir Petrov2, James Glattfelder3, and Richard Olsen4

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2Department of Banking and Finance, University of Zurich
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Abstract

We present an analytic relationship that connects scaling laws in physical and directional-change intrinsic time, namely squared price changes, number of directional changes and variability of overshoots. We provide guidelines on how to apply the relationship in practice.

Keywords: scaling laws, physical time, intrinsic time.

1 Introduction

Interest in scaling relations in FX data was sparked in 1990 by a seminal paper relating the mean absolute change of the logarithmic mid-prices, sampled at time intervals Δt over a sample of size ΔT, to the size of the time interval (Müller et al., 1990). Later, in 1997, a second scaling law was reported by Guillaume et al. (1997), relating the number of so-called directional changes N(δ) to the size of directional-change threshold δ. Time in financial markets does not flow continuously: over weekends trading comes to a standstill, but at news announcements, there are spurts of market activity. The so-called directional change scaling law (Guillaume et al., 1997) was a first attempt at establishing a new paradigm by looking beyond such constraints of the calendar time within financial data, constituting an event-driven approach, where patterns emerge for successions of events at different magnitudes. This alternative approach defines an activity-based, endogenous time-scale called directional-change intrinsic time. This event-driven paradigm was used to observe new, stable patterns that relate the so-called price overshoots and price reversals (Glattfelder et al., 2011).

In this paper, we present an analytic relationship that connects scaling laws in physical and intrinsic time, namely squared price changes, number of directional changes and variability of overshoots. We provide guidelines on how to apply the relationship in practice to better analyse the evolution and statistical properties of high-frequency time series.

2 Physical and Intrinsic Time

Readers intimately familiar with the concept of directional-change intrinsic time can skip the first part of this paper as it lays the ground for terminology and notation.

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Throughout this paper we will assume that the price process \( (P_t : t \geq 0) \) is governed by Brownian motion \( (W_t : t \geq 0) \) with volatility \( \sigma \)
\[
dP_t = \sigma dW_t. \tag{1}
\]
In practice, one samples the price over discrete time intervals of length \( \Delta t \) hence where mentioned we will assume discrete returns \( r(\Delta t) \) as price difference \( \Delta P = P(t + \Delta t) - P(t) \).

The first discovered scaling law (Müller et al., 1990) relates the returns \( r(\Delta t) \) sampled over time horizon \( \Delta t \) with the time horizon \( \Delta t \)
\[
\langle r(\Delta t) \rangle_2 \sim \Delta t, \tag{2}
\]
where \( \langle x \rangle_2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 \) is the sample average of squared values. For Brownian motion it is trivial to derive that the scaling law holds
\[
\mathbb{E}[r(\Delta t)^2] = \sigma^2 \Delta t. \tag{3}
\]

The intrinsic time dissects the price curve into directional changes of length \( \delta \) and overshoots \( \omega(\delta) \) related to those directional changes. The number of directional changes within a time period \( [0, T] \) will be denoted with \( N(\delta, \sigma, T) \). The seminal paper of Guillaume et al. (1997) established that number of directional changes scales with the directional change threshold \( \delta \) as
\[
N(\delta, \sigma, T) \sim \delta^{-2}. \tag{4}
\]
In Petrov et al. (2018), it was established that for Brownian motion the scaling law is as the following:
\[
\mathbb{E}[N(\delta, \sigma, T)] = \frac{\sigma^2 \cdot T}{\delta^2}, \tag{5}
\]
which is in line with empirical observation (Glattfelder et al., 2011). The next seminal discovery (Glattfelder et al., 2011) established an empirical scaling law that relates the length of the overshoot \( \omega(\delta) \) and the directional change threshold \( \delta \)
\[
\langle \omega(\delta) \rangle \approx \delta, \tag{6}
\]
where \( \langle x \rangle = \frac{1}{n} \sum_{i=1}^{n} x_i \) is the sample average. The so-called overshoot scaling law sparked a plethora of research, with its biggest contribution to formulation of liquidity providing trading strategies (Golub et al., 2017). In Golub et al. (2016), it was established that for Brownian motion overshoots are exponentially distributed with parameter \( \delta \):
\[
\omega(\delta) \sim \text{Exp}(\delta). \tag{7}
\]
The letter directly leads to the trivially obtained scaling law for Brownian motion which is again in line with empirical results:
\[
\mathbb{E}[\omega(\delta)] = \delta. \tag{8}
\]
In Golub et al. (2016), the overshoot lengths \( \omega(\delta) \) and their deviations from the expected value \( \delta \) in particular was identified with illiquidity of the exchange rate. Such risk-management application of directional-change intrinsic time demonstrated that multi-scale analysis of overshoots can correctly identify and predict liquidity shocks.

In the rest of the paper, we aim to establish an analytic relationship of the squared returns \( r(\Delta t)^2 = (P(t + \Delta t) - P(t))^2 = (\sigma W(t + \Delta T) - \sigma W(t))^2 \), with number of directional changes \( N(\delta, \sigma, T) \) and overshoots \( \omega(\delta) \). The relationship will provide guidelines how to connect empirical scaling laws in physical and intrinsic time.
Let us assume we are observing the price over a time horizon \([0, T]\). Intrinsic time, which we denote with \(\tau\), dissects the price into directional changes and overshoots, creating a subordinated process \(\sigma W_\tau^{[0,T]}\). The process records values when the intrinsic time ticks. The subordinated process \(\sigma W_\tau^{[0,T]}\) over time horizon \([0, T]\) can be expressed as follows

\[
\sigma W_\tau^{[0,T]} = \sum_{i=1}^{N(\delta, \sigma, T)} (-1)^i (\omega(\delta) - \delta) .
\]

(9)

In other words, when the subordinated process records its value with intrinsic tick it is a manifest of an overshoot off-setted with a directional change. That is the exact definition of intrinsic time as it ticks when we have a reversal of size \(\delta\) from the price overshoot \(\omega(\delta)\). The factor \((-1)^i\) in the formula is there to acknowledge that upward and downward overshoots alternate. The factor will be in fact irrelevant in our derivation hence when necessary we will drop it. Note that \(\sum_{i=1}^{N(\delta, \sigma, T)} (-1)^i (\omega(\delta) - \delta)\) is a so-called compounded process.

Now we are ready to derive the most important result. First we note that \(E[\sigma W_\tau^{[0,T]}] = 0\) as it is just the expectation of Brownian motion. Therefore, we will have

\[
\text{Var}(\sigma W_\tau^{[0,T]}) = E[|\sigma W_\tau^{[0,T]}|^2].
\]

(10)

An analytic relationship will now follow from a beautiful result in probability theory called Wald’s equality\(^1\) for variance of compounded processes (Wald, 1944, 1945). We write it out directly:

\[
E[|\sigma W_\tau^{[0,T]}|^2] = \text{Var}(\sum_{i=1}^{N(\delta, \sigma, T)} (-1)^i (\omega(\delta) - \delta))
\]

\[
= \text{Var}(\omega(\delta) - \delta) \cdot E[N(\delta, \sigma, T)] + \text{Var}(\omega(\delta) - \delta)^2 \cdot \text{Var}(N(\delta, \sigma, T)).
\]

Empirical results from Glattfelder et al. (2011) as well as analytic derivation from Golub et al. (2016) have shown that \(\langle \omega(\delta) \rangle \approx \delta\). Here we assume \(E[\omega(\delta)] = \delta\) hence finally the analytic relationship equals

\[
E[|\sigma W_\tau^{[0,T]}|^2] = \text{Var}(\omega(\delta)) \cdot E[N(\delta, \sigma, T)].
\]

(11)

connecting the number of directional changes and variability\(^2\) of overshoots. Notice that for the case of Brownian motion the relationship is in fact an equality since

\[
E[|\sigma W_\tau^{[0,T]}|^2] = \sigma^2 \cdot T,
\]

(12)

and

\[
\text{Var}(\omega(\delta)) \cdot E[N(\delta, \sigma, T)] = \delta^2 \sigma^2 \cdot \frac{T}{\delta^2} = \sigma^2 \cdot T.
\]

Finally, we explain how to connect the scaling laws obtained from sample data. First, we note that sampling the price process \(P_t\) during time \([0, T]\) over discrete time interval \(\Delta t\) we obtain the average squared returns \(\langle r(\Delta t) \rangle^2\) which scales with the time interval \(\Delta t\). Counting the number of directional changes of price \(P_t\) during \([0, T]\) yields \(N(\delta, \sigma, T)\) which scales with

\(^1\)In literature it is also referred to as Law of total variance or Blackwell-Girshick equation.

\(^2\)The author has incorrectly claimed for years that squared returns are connected only with the number of directional changes and average overshoot!
$\delta^{-2}$ while obtaining the overshoots we can compute their variability (against the mean overshoot which is approximately $\delta$) $\langle \omega(\delta) - \delta \rangle_2$ which should scale as $\delta^2$. The analytic relationship in [11] then yields that the scaling laws are connected as

$$\frac{T}{\Delta t} \cdot \langle r(\Delta t) \rangle_2 \approx \langle \omega(\delta) - \delta \rangle_2 \cdot N(\delta, \sigma, T).$$

(13)

In other words, the scaling law of average squared price changes is a product of the number of directional changes and variability of overshoots. To the best of the author’s knowledge, this is the first time variability of overshoots, and not the average, is deemed relevant for price analysis in intrinsic time.

References


